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String and M-theory deformations of manifolds with special holonomy

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Abstract: The R^4-type corrections to ten and eleven dimensional supergravity required by string and M-theory imply corrections to supersymmetric supergravity compactifications on manifolds of special holonomy, which deform the metric away from the original holonomy. Nevertheless, in many such cases, including Calabi-Yau compactifications of string theory and G_2-compactifications of M-theory, it has been shown that the deformation preserves supersymmetry because of associated corrections to the supersymmetry transformation rules, Here, we consider Spin(7) compactifications in string theory and M-theory, and a class of non-compact SU(5) backgrounds in M-theory. Supersymmetry survives in all these cases too, despite the fact that the original special holonomy is perturbed into general holonomy in each case.

Keywords: M-Theory, Superstring Vacua.

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Contents

1. Introduction

An important question in superstring theory is whether there are compactifications to a lower-dimensional Minkowski spacetime that preserve some fraction of the supersymmetry of the 10-dimensional Minkowski vacuum. At string tree-level, this question can be addressed within the α' expansion of the effective supergravity theory; we shall consider only type-II string theories for which the leading α' correction occurs at order α'^3 and has an R^4 structure. If fermions are omitted, then the corrected action for the metric and dilaton takes the form

$$\mathcal{L} = \sqrt{-g}\, e^{-2\phi} \left(R + 4(\partial\phi)^2 - c\,\alpha'^3\, Y \right) \tag{1.1}$$

for a known constant c, proportional to $\zeta(3)$, and a known scalar Y that is quartic in the Riemann tensor of the 10-dimensional spacetime.

We shall primarily be concerned with solutions to the equations of motion of this action for which the dilaton is constant to lowest order and the 10-dimensional spacetime is the product of 2-dimensional Minkowski spacetime with some initially Ricci-flat riemannian 8-dimensional manifold M_8, with curvature tensor $R_{ijk\ell}$. In this case,

$$Y = \frac{1}{64}\left(t^{i_1\cdots i_8}\,t^{j_1\cdots j_8} - \frac{1}{4}\epsilon^{i_1\cdots i_8}\,\epsilon^{j_1\cdots j_8}\right) R_{i_1i_2j_1j_2}\,R_{i_3i_4j_3j_4}\,R_{i_5i_6j_5j_6}\,R_{i_7i_8j_7j_8}\,. \tag{1.2}$$
$$\equiv Y_0 - Y_2\,, \tag{1.3}$$

where the SO(8)-invariant t-tensor is defined by

$$t^{i_1\cdots i_8}\,M_{i_1i_2}\ldots M_{i_7i_8} = 24 M_i{}^j\,M_j{}^k\,M_k{}^\ell\,M_\ell{}^i - 6(M_i{}^j\,M_j{}^i)^2 \tag{1.4}$$

for an arbitrary antisymmetric tensor $M_{i_1i_2}$. Note that in the decomposition $Y = Y_0 - Y_2$ in (1.3), the subscripts on Y_0 and Y_2 indicate that these are the terms built with 0 and 2 epsilon tensors respectively.

If M_8 is assumed to be compact, then consistency with the initially nondilatonic structure requires [1, 2]

$$\int_{M_8} Y = \mathcal{O}\left(\alpha'\right). \tag{1.5}$$

The simplest way to satisfy this criterion is to demand that $Y = 0$ to leading order in the α' expansion, and this is satisfied if $M_8 = K_8$ for some manifold K_8 of special holonomy (which is necessarily Ricci-flat). This is also what one needs for the lowest-order solution to preserve supersymmetry. The number of supersymmetries preserved equals the number of linearly-independent Killing spinors; i.e., real SO(8) spinors ψ_0 satisfying

$$R_{ijk\ell}\tilde{\Gamma}^{k\ell}\psi_0 = 0\,. \tag{1.6}$$

where $\tilde{\Gamma}^i$ are the 16×16 real SO(8) Dirac matrices (the notation is chosen to agree with that of [2]).

To see why one has $Y = 0$ when $M_8 = K_8$ of special holonomy, we note that Y can be expressed as a Berezin integral [1]

$$Y \propto \int d^{16}\psi\,\exp\left[\left(\bar{\psi}_-\tilde{\Gamma}^{ij}\,\psi_-\right)\left(\bar{\psi}_+\tilde{\Gamma}^{k\ell}\,\psi_+\right)\,R_{ijk\ell}\right], \tag{1.7}$$

where $\bar{\psi} = \psi^T$ and the integration is over the 16 components of a real anticommuting constant SO(8) spinor or, equivalently, over all 16 linearly-independent SO(8) spinors ψ. We can write $\psi = \psi_+ + \psi_-$, where $\psi_\pm$ are the chiral and antichiral projections of ψ. If there are any Killing spinors amongst them, as there will be if $M_8 = K_8$ of special holonomy, then the rules of Berezin integration imply that $Y = 0$.

If we use α and $\dot{\alpha}$ to denote 8-component right-handed and left-handed spinor indices respectively, then up to an inessential constant factor, (1.7) can be rewritten as

$$\begin{aligned} Y = {}& \epsilon^{\alpha_1\cdots\alpha_8}\,\epsilon^{\dot{\beta}_1\cdots\dot{\beta}_8}\,\Gamma^{i_1i_2}_{\alpha_1\alpha_2}\cdots\Gamma^{i_7i_8}_{\alpha_7\alpha_8}\,\Gamma^{j_1j_2}_{\dot{\beta}_1\dot{\beta}_2}\cdots\Gamma^{j_7j_8}_{\dot{\beta}_7\dot{\beta}_8}\times \\ & \times R_{i_1i_2j_1j_2}\,R_{i_3i_4j_3j_4}\,R_{i_5i_6j_5j_6}\,R_{i_7i_8j_7j_8}\,. \end{aligned} \tag{1.8}$$

It is straightforward to show that

$$\frac{1}{256}\epsilon^{\alpha_1\cdots\alpha_8}\,\Gamma^{i_1i_2}_{\alpha_1\alpha_2}\,\Gamma^{i_3i_3}_{\alpha_3\alpha_3}\,\Gamma^{i_6i_6}_{\alpha_5\alpha_6}\,\Gamma^{i_7i_8}_{\alpha_7\alpha_8} \equiv t_+^{i_1\cdots i_8} = t^{i_1\cdots i_8} + \frac{1}{2}\epsilon^{i_1\cdots i_8}\,, \tag{1.9}$$

$$\frac{1}{256}\epsilon^{\dot\beta_1\cdots\dot\beta_8}\,\Gamma^{j_1j_2}_{\dot\beta_1\dot\beta_2}\,\Gamma^{j_3j_4}_{\dot\beta_3\dot\beta_4}\,\Gamma^{j_5j_6}_{\dot\beta_5\dot\beta_6}\,\Gamma^{j_7j_8}_{\dot\beta_7\dot\beta_8} \equiv t_-^{i_1\cdots i_8} = t^{j_1\cdots j_8} - \frac{1}{2}\epsilon^{j_1\cdots j_8}\,, \tag{1.10}$$

(with one overall convention choice determining which right-hand side has the plus sign, and which the minus). Thus we see that (1.7) is of the form $t_-\, t_+\, R^4$, and hence gives rise to (1.2).

It might appear from this result that configurations with constant dilaton and a space-time of the form $\mathbb{E}^{(1,1)} \times K_8$ will automatically continue to be solutions of the α'^3-corrected field equations, in which case one would expect the special holonomy of K_8 to guarantee that supersymmetry is preserved. This is true if $K_8 = T^4 \times K_4$ for a 4-manifold of SU(2) holonomy (i.e., a hyper-Kähler manifold) but it is false in general because, although Y vanishes for a manifold of special holonomy, its *variation* with respect to the metric yields a tensor X_{ij} as a source in the corrected Einstein equations, and this tensor may be non-zero even though $Y = 0$. Specifically, under the circumstances described, the corrected Einstein and dilaton equations are to $\mathcal{O}(\alpha'^3)$

$$\begin{aligned} R_{ij} + 2\nabla_i\nabla_j\,\phi &= c\,\alpha'^3\, X_{ij}\,,\\ R + 4\nabla^2\phi &= 0\,. \end{aligned} \tag{1.11}$$

When $K_8 = T^2 \times K_6$ for a 6-dimensional manifold of SU(3) holonomy; i.e., when K_6 is a Calabi-Yau (CY) manifold, the correction due to the tensor X_{ij} deforms the leading-order CY metric to one of U(3) holonomy [3]. However, as shown in [4], this deformation does not break the supersymmetry of the undeformed solution because there is a compensating α'^3 correction to the gravitino supersymmetry transformation law or, equivalently, to the covariant derivative acting on spinors. More precisely, it was shown that there is a *possible* corrected covariant derivative that has this property; it is expected that this will be needed for a construction of the supersymmetric extension of the lagrangian (1.1), but this complete construction has yet to be carried out in sufficient detail. Nevertheless, this perspective makes it clear that any proposed corrections to the supersymmetry transformations must be expressible in purely riemannian terms, *without* the use of any special structures arising from special holonomy. The proposal of [4] passes this test, which is quite non-trivial in view of the fact that the methods used (details of which can be found in the review [5]) relie heavily on the Kähler properties of CY manifolds. It turns out that the purely riemannian form of the corrected covariant derivative has an obvious extension to 8-manifolds, which is all that will be needed here, and the result can then be summarised by saying that the standard covariant derivative ∇_i acting on SO(8) spinors must be replaced by

$$\hat\nabla_i = \nabla_i - \frac{3c}{4}\alpha'^3\,\left[(\nabla^j\, R_{ikm_1m_2})\, R_{j\ell m_3m_4}\, R^{k\ell}{}_{m_5m_6}\right]\,\tilde\Gamma^{m_1\cdots m_6} + \mathcal{O}\left(\alpha'^4\right) \tag{1.12}$$

It is important to appreciate that it was *not* claimed in [4] that this is the only correction of relevance to this order in the α' expansion, but rather that this term is sufficient for lowest-order backgrounds of the form $\mathbb{E}^{(1,3)} \times K_6$ and its related toroidal compactifications such as $\mathbb{E}^{(1,1)} \times T^2 \times K_6$. In particular, there could be additional terms that are non-zero for a spacetime of the form $\mathbb{E}^{(1,1)} \times M_8$ but which vanish when $M_8 = T^2 \times K_6$.

It is also important to appreciate that the question of whether or not the special-holonomy backgrounds continue to be supersymmetric in the face of α'^3 corrections is one that cannot be addressed unless one has knowledge of the order α'^3 correction to the gravitino transformation rule.[1] At perturbation orders higher than α'^3, there will also certainly be further corrections. In the present paper, however, we limit the discussion to at most this order.

Similar issues arise when $K_8 = S^1 \times K_7$ for a 7-manifold K_7 of G_2 holonomy, as one would expect since the special case $K_7 = S^1 \times K_6$ yields $K_8 = T^2 \times K_6$. In particular, the $\alpha'^3 R^4$ corrections to supergravity arising from the exchange of massive string states must deform any lowest-order compactification on a manifold of initial G_2 holonomy to a compactification on a manifold of generic SO(7) holonomy, and it is far from obvious that such a solution will continue to preserve supersymmetry. Moreover, as G_2-manifolds are not Kähler, the methods used to address this issue in the CY case are no longer available. However, using the existence of the associative 3-form on a G_2 manifold, we were able to show in a previous paper [2] that there is a simple correction to the covariant derivative on spinors that implies supersymmetry preservation of the modified solution, and we used this result to determine the explicit form of the correction for most of the known classes of cohomogeneity-one 7-metrics with G_2 structures (as was done for an analogous class of CY metrics in [7]). Despite the fact that our simple form of the corrected covariant derivative made explicit use of the associative 3-form available only for G_2 manifolds, it was again found possible (by making crucial use of properties of G_2 manifolds) to rewrite this corrected covariant derivative in purely riemannian terms. There is again an obvious extension to 8-manifolds, and the resulting covariant derivative acting on SO(8) spinors was once again found to be (1.12).

Corrections to the effective supergravity action of the form R^4 arise not only at tree level in string theory but also at the one-loop level. This correction is related by dualities to an analogous R^4 M-theory correction to 11-dimensional supergravity. The latter has a structure that differs from the R^4 tree-level string-theory correction, and it also includes an $A \wedge X_8$ Chern-Simons (CS) term that is absent at tree level in string theory. However, for G_2 compactifications, these differences are unimportant, so we were able to lift our string-theory results directly to M-theory. There was a subtlety, however, arising from the fact that an α'^3 correction to the dilaton was needed at tree-level in string theory whereas there is no dilaton in 11 dimensions. However, the effect of the dilaton in string theory can be achieved in M-theory by a modification of the R^4 invariant via a field redefinition. We were thus able to show (i) that M-theory implies a modification of G_2 compactifications of 11-dimensional supergravity in which the 7-metric of G_2 holonomy is

[1] After the completion of the first version of this paper, but before its submission to the archives, there appeared a paper [6] having some overlap with our Spin(7) results, but without any discussion of the order α'^3 corrections to the supersymmetry transformation rules that are needed to address the question that is central here; i.e., whether supersymmetry is maintained in the corrected background.

deformed to one of generic, SO(7), holonomy, and (ii) that ($\mathcal{N} = 1$) supersymmetry of the effective four-dimensional theory is maintained, despite this deformation, at least to order α'^3.

One purpose of this paper is to extend our results on G_2 compactifications, as summarised above, to Spin(7) compactifications. In this respect this should be considered as a companion paper to [2]. At tree-level in string theory our Spin(7) results are similar to those obtained in [2], although there are some additional technical difficulties and subtleties. We also determine explicit supersymmetry-preserving α'^3 corrections for some of the known classes of cohomogeneity-one 8-metrics with Spin(7) structures. At one-loop in string theory, or in M-theory, however, there are more substantial differences arising from the necessity to take into account the Chern-Simons terms associated with the R^4 corrections, and for compact K_8 there is also a topological constraint that must be taken into account. We find that there is nevertheless a supersymmetric deformation of Spin(7) compactifications of M-theory, and hence of 1-loop corrected IIA superstring theory, whether or not the Spin(7) manifold is actually compact.

Another purpose of this paper is to consider the effects of the R^4 corrections of M-theory on compactifications of eleven-dimensional supergravity on ten-manifolds of SU(5) holonomy. This is of considerable interest because it probes aspects of M-theory lying beyond those that are accessible from perturbative string theory. We find corrections to the leading-order backgrounds, and we also consider their supersymmetry. As for the Spin(7) compactifications, there is a topological constraint to take into account. This constraint arises for any SU(5)-holonomy 10-manifold K_{10} with non-trivial homology group H_8. As H_8 is isomorphic to H_2 for any compact 10-manifold (by Poincaré duality) and since H_2 is obviously non-trivial (because K_{10} is Kähler), there is a topological constraint on SU(5)-holonomy compactifications of M-theory, and this constraint even applies to non-compact backgrounds if H_8 is non-trivial. The implications for supersymmetry of of this topological constraint are not at present fully clear to us, so we shall restrict ourselves here to the class of non-compact 10-manifolds K_{10} of SU(5) holonomy for which H_8 is trivial and for which the topological constraint is therefore trivially satisfied. Even so, our results for this case are worthy of note; we find that the same correction to the gravitino transformation rule that ensured the continued supersymmetry of the Spin(7) holonomy backgrounds also implies that the corrected SU(5) holonomy backgrounds maintain supersymmetry. Interestingly, the corrected SU(5) background is no longer even Kähler, but it is still a complex manifold of vanishing first Chern class.

2. Spin(7) preliminaries

As pointed out in [2], the structure of the R^4 invariant Y implies that the tensor X_{ij}, which arises from the variation of Y and which appears in the corrected Einstein field equation, takes the form

$$X_{ij} = \tilde{X}_{ij} + \nabla^k \nabla^\ell X_{ijk\ell} \tag{2.1}$$

for a tensor $\tilde{X}_{ij}$, quartic in the curvatures and a tensor $X_{ijk\ell}$ that is cubic in curvatures. We will show in this section that if the variational expression X_{ij} is then evaluated in a background that has Spin(7) holonomy, then

$$\tilde{X}_{ij} = 0 \tag{2.2}$$

and in fact X_{ij} is given by

$$X_{ij} = \frac{1}{2} c^{mnk}{}_{(i}\, c^{pq\ell}{}_{j)}\, \nabla_k\, \nabla_\ell\, Z_{mnpq} + \nabla^k\, \nabla_\ell\, Z_{mnk(i}\, c^{mn\ell}{}_{j)}\,, \tag{2.3}$$

where $c_{ijk\ell}$ is the calibrating 4-form on the Spin(7) holonomy background, and

$$Z^{mnpq} = \frac{1}{64}\, \epsilon^{mni_1\cdots i_6}\, \epsilon^{pqj_1\cdots j_6}\, R_{i_1i_2j_1j_2}\, R_{i_3i_4j_3j_4}\, R_{i_5i_6j_5j_6}\,. \tag{2.4}$$

2.1 Properties of Spin(7) manifolds

We begin with some basic results about Spin(7) manifolds. There is a single real (commuting) Killing spinor, η, that is either chiral or anti-chiral. We choose conventions in which η is anti-chiral, corresponding to the Spin(7) decomposition

$$8_+ \longrightarrow 8\,, \qquad 8_- \longrightarrow 7+1\,. \tag{2.5}$$

of the chiral/anti-chiral spinor irreps of SO(8). Note that the vector representation of SO(8) remains irreducible:

$$8_v \longrightarrow 8\,. \tag{2.6}$$

We shall normalise the commuting Killing spinor η so that $\bar\eta\,\eta = 1$ (where $\bar\eta = \eta^T$). Given this normalisation, and introducing Γ_9 as the (real) SO(8) chirality matrix, we have the identities

$$\Gamma_i\,\eta\,\bar\eta\,\Gamma^i = 1\!\!1_+\,, \qquad \eta\,\bar\eta - \frac{1}{8}\Gamma_{ij}\,\eta\,\bar\eta\,\Gamma^{ij} = 1\!\!1_- \tag{2.7}$$

where

$$1\!\!1_\pm \equiv \frac{1}{2}(1\!\!1 \pm \Gamma_9)\,, \tag{2.8}$$

which is the identity operator projected into the chiral or anti-chiral spin bundle.

The calibrating 4-form has components that are expressible as

$$c_{ijk\ell} = \bar\eta\,\Gamma_{ijk\ell}\,\eta\,. \tag{2.9}$$

It is straightforward to establish the following identities:

$$c_{ijk\ell}\,\Gamma^{k\ell}\,\eta = -6\Gamma_{ij}\,\eta\,, \tag{2.10}$$

$$c_{ijkp}\,c^{\ell mnp} = 6\delta^{\ell mn}_{ijk} - 36\delta_{[i}{}^{[\ell}\,c_{jk]}{}^{mn]}\,. \tag{2.11}$$

Recalling the Killing spinor integrability condition $R_{ijk\ell}\tilde\Gamma^{k\ell}\eta = 0$, one can also show that

$$R_{ijk\ell}\,c^{k\ell}{}_{mn} = 2R_{ijmn}\,; \tag{2.12}$$

this is the condition for Spin(7) holonomy.

2.2 Correction to the Einstein equations

In order to derive the α'^3 corrections to the Einstein equations at string tree level, we need to evaluate the variation of the quartic-curvature term Y. This was relatively straightforward in the case of corrections to six-dimensional Calabi-Yau backgrounds K_6, [3], and for corrections to seven-dimensional G_2-holonomy backgrounds K_7 [2]. The reason for this is that in each case, one has SO(8) Killing spinors of both chiralities in the $K_8 = \mathbb{R}^2 \times K_6$, or $K_8 = \mathbb{R} \times K_7$ eight-dimensional transverse space. This means that when one varies the metrics or vielbeins in the Berezin integral (1.7), the only terms that can survive are those where the metrics in one of the Riemann tensors itself are varied. This is because they are the only terms where one does not inevitably end up with Killing spinors linked to an unvaried Riemann tensor and thus vanishing by virtue of (1.6).

Additionally, because of the non-chiral nature of the Killing spinors in K_8 in the previous cases, it was straightforward to express the variation of Y, originally written in terms of spinors in the Berezin integral (1.7), in terms of tensorial quantities built from Riemann tensors and the Kähler form of K_6 or the associative 3-form of K_7. This stemmed from the fact that for both chiral and antichiral SO(8) spinors, one had decompositions under SU(3) or G_2 that provided a one-to-one mapping between the vector and the spinor representation in K_6 or K_7.

In the case of Spin(7) holonomy manifolds K_8 things are more subtle for two reasons. Firstly, we have a Killing spinor of only one eight-dimensional chirality, which we are taking, by convention choice, to be antichiral. This means that we could, *a priori*, encounter non-vanishing terms in the variation of Y, defined in (1.7), in which vielbeins used in contracting the Riemann tensors onto the Dirac matrices are varied, leaving all four Riemann tensors unvaried.

Secondly, we can see from (2.5) and (2.6) that, while the 8_+ spinor representation of SO(8) is indeed isomorphic to the 8_v vector representation in a Spin(7) background, the 8_- spinor representation is not. This could lead to obstacles in rewriting the variation of Y, given by (1.7), in a purely tensorial form.

To address these problems, it is helpful to introduce two further quartic-curvature invariants, which we shall call Y_- and Y_+. These are defined in terms of Berezin integrals analogous to (1.7), except that now we have

$$Y_+ \propto \int d^8\psi_+\, d^8\chi_+ \exp\Big[\Big(\bar\psi_+\tilde\Gamma^{ij}\,\psi_+\Big)\Big(\bar\chi_+\tilde\Gamma^{k\ell}\,\chi_+\Big)\,R_{ijk\ell}\Big]\,. \tag{2.13}$$

$$Y_- \propto \int d^8\psi_-\, d^8\chi_- \exp\Big[\Big(\bar\psi_-\tilde\Gamma^{ij}\,\psi_-\Big)\Big(\bar\chi_-\tilde\Gamma^{k\ell}\,\chi_-\Big)\,R_{ijk\ell}\Big]\,, \tag{2.14}$$

The integration in (2.13) is over two independent sets of chiral SO(8) spinors, while in (2.14) it is over independent two sets of antichiral spinors. From (1.9) and (1.10), we see that Y_+ and Y_- are given by

$$\begin{aligned} Y_\pm &= \frac{1}{64} t_\pm^{i_1\cdots i_8}\, t_\pm^{j_1\cdots j_8} R_{i_1 i_2 j_1 j_2}\cdots R_{i_7 i_8 j_7 j_8}\,,\\ &= \frac{1}{64}\left(t^{i_1\cdots i_8}\, t^{j_1\cdots j_8} \pm t^{i_1\cdots i_8}\,\epsilon^{j_1\cdots j_8} + \frac{1}{4}\epsilon^{i_1\cdots i_8}\,\epsilon^{j_1\cdots j_8}\right) R_{i_1 i_2 j_1 j_2}\cdots R_{i_7 i_8 j_7 j_8}\,, \end{aligned} \tag{2.15}$$

$$\equiv Y_0 \pm Y_1 + Y_2 \,, \tag{2.16}$$

where Y_0 and Y_2 are the same as in (1.3), and Y_1 is the term in (2.15) that is linear in the epsilon tensor.

A crucial property of the invariants $Y_\pm$ is that they differ from the actual effective action contribution Y by terms that are purely topological in $D = 8$:

$$Y_\pm - Y = \pm Y_1 + 2Y_2 = \left(\pm t^{i_1\cdots i_8} + \frac{1}{2}\epsilon^{i_1\cdots i_8}\right)\epsilon^{j_1\cdots j_8}\, R_{i_1 i_2 j_1 j_2}\cdots R_{i_7 i_8 j_7 j_8}\,. \tag{2.17}$$

As an 8-form, written in terms of the curvature 2-forms Θ_{ij}, the difference is given by

$$*(Y_\pm - Y) = 16\left(\pm t^{i_1\cdots i_8} + \frac{1}{2}\epsilon^{i_1\cdots i_8}\right)\Theta_{i_1 i_2}\wedge\cdots\wedge\Theta_{i_7 i_8}\,, \tag{2.18}$$

which makes the topological nature manifest. Because of this, the integrals of Y, Y_+ and Y_- all have the same variation,[2] evaluated on an eight-dimensional curved background, and so we can use either Y_+ or Y_- in place of Y for the purpose of computing the variation X_{ij} (even though Y_+ does not vanish in the special-holonomy background).[3]

Each of the $Y_\pm$ has its own advantages and disadvantages, when used in place of Y to calculate the variation X_{ij}. If we vary Y_-, then it is manifest that no terms from the variation of the bare vielbeins contracting Riemann tensors $R^\mu{}_{\nu\rho\sigma}$ onto Dirac matrices Γ^{ij} will survive in the Berezin integration. This is because we will always have a contribution either of the form $R_{ijk\ell}\Gamma^{k\ell}\eta$ or $\Gamma^{ij}\eta\, R_{ijk\ell}$ in every term where the explicit vielbeins are varied, and these then vanish by virtue of the integrability condition for the (antichiral) Killing spinor. Thus only terms arising from the variation of metrics contained within the connections from which $R^\mu{}_{\nu\rho\sigma}$ is composed will survive. This means that, after integration by parts, the variation of Y_- will necessarily involve only terms constructed from two covariant derivatives acting on (Riemann)^3 structures, and that there will be no terms quartic in Riemann tensors without derivatives. The drawback to using Y_-, however, is that there is no isomorphism between the decompositions of the 8_- and 8_v representations of SO(8) under restriction to Spin(7), and therefore we do not have a simple direct way of re-expressing δY_- in purely bosonic tensorial terms.

On the other hand, if we vary Y_+ then the isomorphism between the irreducible 8_+ and 8_v representations of SO(8) under restriction to Spin(7) does in this case provide us

[2]To be precise, when we say that Y, Y_+ and Y_- all have the same variation, we mean that their variations differ by total derivatives. At string tree level, where these quantities are multiplied by $e^{-2\phi}$, the total derivatives will integrate by parts to give contributions involving derivatives of the dilaton when comparing the corrected Einstein equations. However, since the Y term is accompanied by an explicit α'^3 factor, and since the dilaton is constant in the leading-order background these extra derivative terms contribute at best at order α'^6 in the corrected Einstein equations, and thus they may be neglected at the α'^3 order to which we are working. At string one-loop, or in M-theory, there is no dilaton prefactor, and so the integration by parts simply gives zero.

[3]We should note, because a failure to do so has caused some confusion in the earlier literature, that the computed result for the Berezin integral for Y that is given in ref. [1] is actually the result obtained by computing the Berezin integral for Y_+, but since $Y_+ = Y$ for the CY compactifications considered there, and since the variations are also the same, the distinction was unimportant there. In our case, however, the distinction is important.

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with a simple way to recast δY_+ in purely bosonic tensorial terms. The drawback to using Y_+, however, is that there are no spinor zero modes at all in the Berezin integral (2.13), and so it is not immediately manifest that the terms coming from the variation of the bare vielbeins that contract Riemann tensors $R^\mu{}_{\nu\rho\sigma}$ onto Dirac matrices Γ^{ij} will not contribute. Indeed, Y_+ itself does not even vanish in the Spin(7) background.

We can however make use of the complementary properties that are manifested in the different expressions Y, Y_+ and Y_-, and thereby "have our cake and eat it too." In particular, we note that the difference $Y_+ - Y_-$ is also topological,

$$Y_+ - Y_- = 2t^{i_1\cdots i_8}\,\epsilon^{j_1\cdots j_8}\,R_{i_1i_2j_1j_2}\cdots R_{i_7i_8j_7j_8}\,, \tag{2.19}$$

which means that after the varied expression is specialised to a Spin(7) background, we must have it that δY_- and δY_+ give the same contribution to the corrected Einstein equations, at order α'^3. In particular, we can see that (2.19) may be written in terms of Riemann tensors $R^\mu{}_{\nu\rho\sigma}$ without the use of any bare metrics or vielbeins. We can now invoke the above observation that the variation of Y_- does not contain any terms coming from the variation of bare vielbeins to see that there will be no such terms in the variation of Y_+ either. Then, we are in a position to exploit the isomorphism between the decompositions of the 8_+ and 8_v representations of SO(8) under restriction to Spin(7) to obtain a simple tensorial expression for δY_+, and hence δY.

It follows from (1.9) and (2.15) that we shall have

$$\begin{aligned} Y_+ \propto\; & \epsilon^{\alpha_1\cdots\alpha_8}\,\epsilon^{\beta_1\cdots\beta_8}\,\Gamma^{i_1i_2}_{\alpha_1\alpha_2}\cdots\Gamma^{i_7i_8}_{\alpha_7\alpha_8}\,\Gamma^{j_1j_2}_{\beta_1\beta_2}\cdots\Gamma^{j_7j_8}_{\beta_7\beta_8}\times \\ & \times R_{i_1i_2j_1j_2}\,R_{i_3i_4j_3j_4}\,R_{i_5i_6j_5j_6}\,R_{i_7i_8j_7j_8}\,. \end{aligned} \tag{2.20}$$

Because the 8_+ and 8_v representation become the same irreducible representation of Spin(7), the expression (2.20) can be rewritten such that only vector indices are needed. Specifically, the mapping between 8_+ and 8_v is implemented by

$$\nu^i_\alpha = \Gamma^i{}_{\alpha\dot\beta}\,\eta^{\dot\beta}\,. \tag{2.21}$$

This matrix has unit determinant, and so we can write

$$\epsilon^{\alpha_1\cdots\alpha_8} = \nu^{\alpha_1}_{i_1}\cdots\nu^{\alpha_8}_{i_8}\,\epsilon^{i_1\cdots i_8}\,. \tag{2.22}$$

Since we have argued that there will be no contributions coming from varying the bare vielbeins in (2.20), after specialising the varied expression to a Spin(7) background, we need only vary the metrics in the connections from which the Riemann tensors themselves are constructed. Up to a constant factor, which is as yet inessential to our discussion, we therefore have

$$\begin{aligned} \delta Y_+ =\; & 4\epsilon^{\alpha_1\cdots\alpha_8}\,\epsilon^{\beta_1\cdots\beta_8}\,(\Gamma^{i_1i_2})_{\alpha_1\alpha_2}\cdots(\Gamma^{i_7i_8})_{\alpha_7\alpha_8}\,(\Gamma^{j_1j_2})_{\beta_1\beta_2}\cdots(\Gamma^{j_7j_8})_{\beta_7\beta_8}\times \\ & \times R_{i_1i_2j_1j_2}\cdots R_{i_5i_6j_5j_6}\,\delta R_{i_7i_8j_7j_8}\,, \\ =\; & 8\epsilon^{\alpha_1\cdots\alpha_8}\,\epsilon^{\beta_1\cdots\beta_8}\,(\Gamma^{i_1i_2})_{\alpha_1\alpha_2}\cdots(\Gamma^{i_7i_8})_{\alpha_7\alpha_8}\,(\Gamma^{j_1j_2})_{\beta_1\beta_2}\cdots(\Gamma^{j_7j_8})_{\beta_7\beta_8}\times \\ & \times R_{i_1i_2j_1j_2}\cdots R_{i_5i_6j_5j_6}\,\nabla_{i_7}\nabla_{j_7}\,\delta g_{i_8j_8}\,. \end{aligned} \tag{2.23}$$

where $\delta R_{i_7i_8j_7j_8}$ denotes the variation of the Riemann tensor with respect to the metric.

From the properties (2.10) and (2.11), one easily shows that

$$\bar\eta\,\Gamma_i\,\Gamma^{k\ell}\,\Gamma_j\,\eta = c_{ij}{}^{k\ell} + 2\delta_{ij}^{k\ell}\,, \tag{2.24}$$

and hence, using (2.12) repeatedly, we see that up to a further inessential overall factor (and specialised to the Spin(7) background) we have

$$\delta Y_+ = Z^{mnpq}\,(c_{mn}{}^{ij} + 2\delta^{ij}_{mn})\,(c_{pq}{}^{k\ell} + 2\delta^{k\ell}_{pq})\,\delta R_{ijk\ell} \tag{2.25}$$

where

$$Z^{mnpq} = \frac{1}{64}\,\epsilon^{mni_1\cdots i_6}\,\epsilon^{pqj_1\cdots j_6}\,R_{i_1i_2j_1j_2}\,R_{i_3i_4j_3j_4}\,R_{i_5i_6j_5j_6}\,. \tag{2.26}$$

The following useful properties of Z^{mnpq} can easily be established:

$$\begin{aligned} Z^{mnpq} = Z^{pqmn} &= -Z^{nmpq} = -Z^{mnqp}\,,\\ \nabla_m\,Z^{mnpq} &= 0\,,\qquad c_{mnpr}\,Z^{mnpq} = 0\,. \end{aligned} \tag{2.27}$$

We therefore conclude that the variation of Y gives

$$X_{ij} = \frac{1}{2} c^{mnk}{}_{(i}\,c^{pq\ell}{}_{j)}\,\nabla_k\,\nabla_\ell\,Z_{mnpq} + \nabla^k\,\nabla_\ell\,Z_{mnk(i}\,c^{mn\ell}{}_{j)}\,. \tag{2.28}$$

Note that a simple calculation using (2.10), (2.11), (2.12) and (2.27) shows that

$$g^{ij}\,X_{ij} = \Box Z\,, \tag{2.29}$$

and hence from (1.11) we learn that

$$R_{ij} = c\,\alpha'^3\,(X_{ij} + \nabla_i\nabla_j\,Z)\,, \tag{2.30}$$

$$\phi = -\frac{1}{2}c\,\alpha'^3\,Z\,. \tag{2.31}$$

3. Correction to the supersymmetry transformation rule

Since the effect of the α'^3 corrections is to deform the original Spin(7) metric to one that is no longer Ricci flat, it follows that it will no longer have Spin(7) holonomy and so it will no longer admit a covariantly constant spinor. However, one knows that at the same time as the α'^3 corrections to the string effective action set in, there also will be corresponding corrections to the supersymmetry transformation rules at the α'^3 order. These were discussed in the context of six-dimensional Calabi-Yau backgrounds in refs. [4, 5], where it was indeed shown that the deformed metrics, which acquire an extra U(1) factor to their original undeformed SU(3) holonomy, have the feature of still admitting spinors that are constant with respect to a modified covariant derivative. This $O(\alpha'^3)$ modification can be understood as the necessary correction to the gravitino transformation rule at this order. This issue was discussed further for Calabi-Yau backgrounds in [7], and for seven-dimensional backgrounds with G_2 holonomy in [2].[4]

[4]It should be emphasised that if these order α'^3 corrections to the supersymmetry transformation rule are not included, then one will not be able to demonstrate the preservation of supersymmetry in the α'^3-corrected backgrounds.

Here, we shall begin by introducing the following modified covariant derivative,[5]

$$D_i \equiv \nabla_i + Q_i = \nabla_i + \frac{1}{4} c\, \alpha'^3\, c_{ijk\ell}\, \nabla^j\, Z^{k\ell mn}\, \Gamma_{mn}\,. \tag{3.1}$$

where the Z-tensor is the one defined in (2.26). After some algebra, which involves making extensive use of properties given in subsection 2.1, one finds that the integrability condition $[D_i, D_j]\,\eta = 0$ for the existence of a spinor satisfying $D_i\,\eta = 0$ precisely implies that (2.30) holds. This, therefore, is our candidate expression for the modification to the gravitino transformation rule in an originally Spin(7) background; $\delta\psi_i = D_i\,\epsilon$.

As it stands, (3.1) is written using the special tensor $c_{ijk\ell}$ specific to a Spin(7) background. One knows, of course, that the modified supersymmetry transformation rules (and also the modified equations of motion) should all be expressible in fully covariant riemannian terms, making no use of additional invariant tensors that exist only in special backgrounds. This question has been addressed for Calabi-Yau and G_2 backgrounds in the previous literature [4, 2], and indeed the candidate expressions for the modified supersymmetry transformation rules that were written down in [4, 5] were fully riemannian expressions that were shown to be compatible with special forms written in Kähler language. In [2], it was shown that the riemannian expressions in [4, 5] were also compatible with a special form written using the calibrating 3-form in a G_2 background.

Here, we shall show that the modified derivative D_i defined in (3.1) can be re-expressed without the use of the special tensor $c_{ijk\ell}$ of a Spin(7) background, and that in fact (3.1) is nothing but the Spin(7) specialisation of the riemannian results conjectured in refs. [4, 5].

To do this, it is useful first to note that we have

$$\begin{aligned} c_{ijk\ell}\, \epsilon^{k\ell i_1\cdots i_6} &= \bar\eta\, \Gamma_{ijk}\, \Gamma_\ell\, \eta\, \epsilon^{k\ell i_1\cdots i_6} = \bar\eta\, \Gamma_{ijk}\, \Gamma^{k i_i\cdots i_6}\eta\,, \\ &= -4\delta_{ij}^{[i_1 i_2}\, c^{i_3 i_4 i_5 i_6]}\,, \end{aligned} \tag{3.2}$$

and hence

$$Q_i = \frac{1}{64} c\, \alpha'^3\, \delta_{ij}^{[i_1 i_2}\, c^{i_3 i_4 i_5 i_6]}\, \epsilon^{mn j_1\cdots j_6}\, \nabla^j\, (R_{i_1 i_2 j_1 j_2}\, R_{i_3 i_4 j_3 j_4}\, R_{i_5 i_6 j_5 j_6})\, \Gamma_{mn} \tag{3.3}$$

Since all the permutations of the indices $\{i_1 \cdots i_6\}$ involve at least one of the Riemann tensors having a double contraction with $c_{ijk\ell}$, it follows that we can make use (2.12) and thereby absorb all occurrences of this special tensor. After performing the necessary combinatoric manipulations, and some further simplifications using the Bianchi identity for the Riemann tensor, we arrive at the result

$$Q_i = -\frac{3}{4} c\, \alpha'^3\, (\nabla^j\, R_{ikm_1m_2})\, R_{j\ell m_3 m_4}\, R^{k\ell}{}_{m_5 m_6}\, \Gamma^{m_1\cdots m_6}\,. \tag{3.4}$$

In this form, Q_i can be recognised as precisely the same modification to the Killing spinor condition that was proposed in [4]. In that case, the proposal was based on a consideration

[5] Note that for our present purposes, where we are simply concerned with establishing the circumstances under which a Killing spinor exists, we may view two formulations of a gravitino transformation rule as equivalent if they agree when acting on the putative Killing spinor. Here, as in much of the previous literature, we shall commonly adopt this viewpoint.

of deformations from SU(3) holonomy for six-dimensional Calabi-Yau backgrounds. It was also shown in [2] that the more stringent conditions arising for G_2 backgrounds lead to exactly the same modification to the Killing spinor condition. Here, we have shown that the yet more stringent conditions of a Spin(7) background again yield the same result, confirming the validity of the riemannian expression (3.4) that was conjectured in [4].

Of course since a six-dimensional space of SU(3) holonomy (times a line or circle) is just a special case of a G_2 manifold, and a seven-dimensional space of G_2 holonomy (times a line or circle) is a special case of a Spin(7) manifold, it follows that our derivation here encompasses the previous SU(3) and G_2 results in [4] and [2].

4. α'^3 corrections for eight-dimensional Kähler metrics

An eight-dimensional Ricci-flat Kähler metric is a Spin(7) metric, since its SU(4) holonomy is contained within Spin(7). Specifically, the embedding can be seen by examining the decomposition of the three eight-dimensional representations of the SO(8) tangent-space group first to Spin(7) and then to SU(4):

SO(8)		Spin(7)		SU(4)
8_+	$\longrightarrow$	8	$\longrightarrow$	$4+\overline{4}$
8_-	$\longrightarrow$	$7+1$	$\longrightarrow$	$6+1+1$
8_v	$\longrightarrow$	8	$\longrightarrow$	$4+\overline{4}$

The two singlets in the decomposition of the 8_- under SU(4) indicate that there are two covariantly-constant left-handed Majorana-Weyl spinors, say η_1 and η_2, in the SU(4)-holonomy metric, which we may normalise to $\bar\eta_A\,\eta_B = \delta_{AB}$. From these, we may define complex left-handed spinors $\eta_\pm$ and $\bar\eta_\pm$ as

$$\eta_\pm \equiv \frac{1}{\sqrt{2}}\,(\eta_1 \pm \mathrm{i}\,\eta_2)\,, \qquad \bar\eta_\pm \equiv \frac{1}{\sqrt{2}}\,(\bar\eta_1 \pm \mathrm{i}\,\bar\eta_2)\,. \tag{4.1}$$

We shall then have

$$\begin{aligned} J_{ij} &= \mathrm{i}\,\bar\eta_+\,\Gamma_{ij}\,\eta_- = \bar\eta_1\,\Gamma_{ij}\,\eta_2\,, & 3J_{[ij}\,J_{k\ell]} &= \bar\eta_+\,\Gamma_{ijk\ell}\,\eta_-\,,\\ \Omega_{ijk\ell} &= \bar\eta_+\,\Gamma_{ijk\ell}\,\eta_+\,, & \overline\Omega_{ijk\ell} &= \bar\eta_-\,\Gamma_{ijk\ell}\,\eta_-\,, \end{aligned} \tag{4.2}$$

where J_{ij} is the Kähler form, and $\Omega_{ijk\ell}$ is the holomorphic 4-form, with its complex conjugate $\overline\Omega_{ijk\ell}$.

We may take the calibrating 4-form $c_{ijk\ell}$ of the SU(4) metric, viewed as a Spin(7) metric, to be given by $c_{ijk\ell} = \bar\eta_1\,\Gamma_{ijk\ell}\,\eta_1$. It then follows from (4.2) that we shall have

$$c_{ijk\ell} = \frac{1}{2}(\Omega_{ijk\ell} + \overline\Omega_{ijk\ell}) + 3J_{[ij}\,J_{k\ell]}\,. \tag{4.3}$$

In a Kähler metric, the only non-vanishing components of the Riemann tensor are "mixed" on both the first index-pair and the second index-pair. In other words if the i index on $R_{ijk\ell}$ is holomorphic then j must be antiholomorphic, and *vice versa*, with a similar property for

k and ℓ. From the definition (2.26) of Z^{mnpq}, it then follows that this tensor must similarly be mixed on its mn indices and in its pq indices. From this, it follows that

$$\Omega_{ijmn}\, Z^{mnpq} = 0\,, \tag{4.4}$$

together with similar relations following from symmetries and from conjugation. A Kähler metric also has the property that

$$J_k{}^m\, J_\ell{}^n\, R_{ijmn} = R_{ijk\ell}\,, \tag{4.5}$$

together with the analogous property on the first index-pair. These expressions can be written more elegantly using the "hat" notation introduce in [8], where, for any vector V_i, one defines

$$V_{\hat{i}} \equiv J_i{}^j\, V_j\,. \tag{4.6}$$

Thus (4.5) becomes $R_{\hat{i}\hat{j}k\ell} = R_{ijk\ell}$. From (2.26), it therefore follows that

$$Z^{\hat{i}\hat{j}pq} = Z^{ijpq}\,, \qquad Z^{mn\hat{i}\hat{j}} = Z^{mnij}\,. \tag{4.7}$$

Using the above results, it is now straightforward to show that the expression for X_{ij} that we obtained for a Spin(7) background in (2.28) reduces to

$$X_{ij} = \frac{1}{2}\nabla_{\hat{i}}\, \nabla_{\hat{j}}\, (J_{mn}\, J_{pq}\, Z^{mnpq}) \tag{4.8}$$

in an eight-dimensional Ricci-flat Kähler background. After a little further manipulation, we find that the result (2.30) for the α'^3 correction to the Ricci-flatness condition in a Spin(7) background reduces for an eight-dimensional Ricci-flat Kähler background to the corrected condition

$$R_{ij} = c\, \alpha'^3\, (\nabla_{\hat{i}}\, \nabla_{\hat{j}} + \nabla_i\, \nabla_j)\, Z\,, \tag{4.9}$$

where, as before, we have defined $Z \equiv Z^{mn}{}_{mn}$. This is in agreement with the standard result that one obtains from the calculation of the supersymmetric sigma-model beta-function at four loops.

In a similar manner, we can specialise the Spin(7) correction term Q_i in the spinor covariant derivative $D_i = \nabla_i + Q_i$ to the case of an eight-dimensional Ricci-flat Kähler metric. Using the properties discussed above, we find that Q_i defined in (3.1) reduces to

$$Q_i = \frac{1}{4} c\, \alpha'^3\, \nabla_{\hat{i}}\, (J_{k\ell}\, Z^{k\ell mn})\, \Gamma_{mn}\,. \tag{4.10}$$

It was shown in [7] that when acting on a covariantly-constant spinor in a Kähler background one has

$$(\Gamma_{ij} + \Gamma_{\hat{i}\hat{j}})\, \eta = 2\mathrm{i}\, J_{ij}\, \eta\,, \tag{4.11}$$

and hence it follows that when acting on η, the modified covariant derivative in the deformed background reduces to

$$\begin{aligned} D_i\, \eta &= \nabla_i\, \eta + \frac{\mathrm{i}}{4}\, c\, \alpha'^3\, \nabla_{\hat{i}}\, (J_{k\ell}\, J_{mn}\, Z^{k\ell mn})\, \eta\,, \\ &= \nabla_i\, \eta + \frac{\mathrm{i}}{2}\, c\, \alpha'^3\, (\nabla_{\hat{i}}\, Z)\, \eta\,. \end{aligned} \tag{4.12}$$

This last expression agrees with the one given in refs. [4, 7].

5. Explicit examples

5.1 S^7 principal orbits

Following [9], we introduce left-invariant 1-forms L_{AB} for the group manifold SO(5). These satisfy $L_{AB} = -L_{BA}$, and

$$dL_{AB} = L_{AC} \wedge L_{CB} \,. \tag{5.1}$$

The 7-sphere is then given by the coset $\mathrm{SO}(5)/\mathrm{SU}(2)_L$, where we take the obvious SO(4) subgroup of SO(5), and write it (locally) as $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$.

If we take the indices A and B in L_{AB} to range over the values $0 \le A \le 4$, and split them as $A = (a, 4)$, with $0 \le a \le 3$, then the SO(4) subgroup is given by L_{ab}. This is decomposed as $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$, with the two sets of SU(2) 1-forms given by the self-dual and anti-self-dual combinations:

$$R_i = \frac{1}{2}\left(L_{0i} + \frac{1}{2}\epsilon_{ijk}\, L_{jk}\right), \qquad L_i = \frac{1}{2}\left(L_{0i} - \frac{1}{2}\epsilon_{ijk}\, L_{jk}\right), \tag{5.2}$$

where $1 \le i \le 3$. Thus the seven 1-forms in the S^7 coset will be

$$P_a \equiv L_{a4}\,, \qquad R_1\,, \qquad R_2\,, \qquad R_3\,. \tag{5.3}$$

The most general cohomogeneity-one metric ansatz for these S^7 principal orbits is

$$ds_8^2 = dt^2 + a_i^2\, R_i^2 + b^2\, P_a^2\,. \tag{5.4}$$

Several complete nonsingular Spin(7) metrics are contained within this class, including the original asymptotically conical (AC) example found in refs. [10, 11], which is uniaxial, $a_1 = a_2 = a_3$, and the family of asymptotically locally conical (ALC) examples found in [12], which are biaxial, with (say) $a_1 = a_2$.

In the natural orthonormal basis for (5.4), namely

$$e^0 = dt\,, \qquad e^i = a_i\, R_i\,, \qquad e^a = b\, P_a\,, \tag{5.5}$$

the calibrating 4-form has components $c_{ijk\ell}$ given by

$$\begin{aligned} 1 &= -c_{0123} = c_{0145} = c_{0167} = c_{0246} = -c_{0257} = c_{0347} = c_{0356}\,, \\ &= c_{1247} = c_{1256} = -c_{1346} = c_{1357} = c_{2345} = c_{2367} = -c_{4567}\,, \end{aligned} \tag{5.6}$$

where we have assigned explicit index values $i = 1, 2, 3$ and $a = 4, 5, 6, 7$. It is now a straightforward mechanical exercise, most easily implemented by computer, to solve first for the covariantly-constant spinor η in the unmodified Spin(7) background, yielding first-order equations for the metric functions a_i and b, and then to find the α'^3-corrected first-order equations that follow from imposing $D_i\,\eta = 0$, where D_i is given in (3.1).[6] The

[6] Note that when we do this, we assume that η retains the identical form that it had in the uncorrected Ricci-flat background. The test of the validity of this assumption is that the corrected first-order equations we obtain under this assumption do indeed imply that the corrected second-order Einstein equations (1.11) are satisfied.

first-order equations in the general triaxial case are rather complicated, and are not easily presentable in this paper. Here, we shall just give our results in the uniaxial special case, where the three metric functions a_i are set equal, $a_i = a$. We then find that a and b must satisfy

$$\frac{\dot a}{a} = \frac{1}{a} - \frac{a}{2b^2} - c\,\alpha'^3\,\dot S_1\,, \qquad \frac{\dot b}{b} = \frac{3a}{4b^2} - c\,\alpha'^3\,\dot S_2\,, \tag{5.7}$$

where c is the usual constant that we introduced in (1.1), and

$$\begin{aligned} S_1 &= \frac{64239a^6 - 227052a^4b^2 + 269712a^2b^4 - 101440b^6}{1064b^{12}}\,,\\ S_2 &= \frac{3(-4389a^6 + 16821a^4b^2 - 20997a^2b^4 + 8756b^6)}{133b^{12}}\,. \end{aligned} \tag{5.8}$$

We can integrate the equations (5.7) to give

$$\begin{aligned} b(r)^2 &= \frac{3}{2}e^{-2c\,\alpha'^3\,\bar S_2(r)}\int^r e^{2c\,\alpha'^3\,\bar S_2(r')}\,dr'\,,\\ a(r)^2 &= 2b(r)^{-\frac{4}{3}}\,e^{-c\,\alpha'^3\,(2\bar S_1(r)+\frac{4}{3}\bar S_2(r))}\int^r b(r')^{\frac{4}{3}}\,e^{c\,\alpha'^3\,(2\bar S_1(r')+\frac{4}{3}\bar S_2(r'))}dr'\,, \end{aligned} \tag{5.9}$$

where the variable r is defined by $dr = a\,dt$ and the bars on S_1 and S_2 denote that these quantities are evaluated in the leading-order background.

5.2 SU(3)/ U(1) principal orbits

The cosets SU(3)/ U(1), known as Aloff-Wallach spaces $N(k,\ell)$, are characterised by two integers k and ℓ, which define the embedding of the U(1) subgroup h of SU(3) matrices according to

$$h = \mathrm{diag}(e^{\mathrm{i}\,k\,\theta}, e^{\mathrm{i}\,\ell\,\theta}, e^{-\mathrm{i}\,(k+\ell)\,\theta})\,. \tag{5.10}$$

If one defines $m = -k-\ell$, it is evident that there is an S_3 symmetry given by the permutations of $(k,\ell,-k-\ell)$.

We define left-invariant 1-forms $L_A{}^B$ for SU(3), where $A = 1,2,3$, $L_A{}^A = 0$, $(L_A{}^B)^\dagger = L_B{}^A$ and $dL_A{}^B = \mathrm{i}\,L_A{}^C\wedge L_C{}^B$, and introduce the combinations

$$\begin{aligned} &\sigma \equiv L_1{}^3\,, \qquad \Sigma \equiv L_2{}^3\,, \qquad \nu \equiv L_1{}^2\,,\\ &\lambda \equiv \sqrt{2}\,\cos\tilde\delta\,L_1{}^1 + \sqrt{2}\,\sin\tilde\delta\,L_2{}^2\,,\\ &Q \equiv -\sqrt{2}\,\sin\tilde\delta\,L_1{}^1 + \sqrt{2}\,\cos\tilde\delta\,L_2{}^2\,, \end{aligned} \tag{5.11}$$

where Q is taken to be the U(1) generator lying outside the SU(3)/ U(1) coset, and

$$\frac{k}{\ell} = -\tan\tilde\delta\,. \tag{5.12}$$

Thus $\tilde\delta$ is restricted to an infinite discrete set of values.

We shall follow [12] and use real left-invariant 1-forms defined by $\sigma = \sigma_1 + \mathrm{i}\,\sigma_2$, $\Sigma = \Sigma_1 + \mathrm{i}\,\Sigma_2$ and $\nu = \nu_1 + \mathrm{i}\,\nu_2$. The cohomogeneity one metrics can then be written as

$$ds_8^2 = dt^2 + a^2\,\sigma_i^2 + b^2\,\Sigma_i^2 + c^2\,\nu_i^2 + f^2\,\lambda^2\,, \tag{5.13}$$

where a, b, c and f are functions of the radial coordinate t. Using the Killing spinor equations that we have derived in this paper, we obtain the first-order equations for this system up to α'^3 order, given by

$$\begin{aligned}
\frac{\dot a}{a} &= \frac{b^2+c^2-a^2}{abc} - \frac{\sqrt{2}\, f\, \cos\tilde\delta}{a^2} - \alpha'^3\, K_1\,,\\
\frac{\dot b}{b} &= \frac{a^2+c^2-b^2}{abc} + \frac{\sqrt{2}\, f\, \cos\tilde\delta}{b^2} - \alpha'^3\, K_2\,,\\
\frac{\dot c}{c} &= \frac{a^2+b^2-c^2}{abc} + \frac{\sqrt{2}\, f\, (\cos\tilde\delta - \sin\tilde\delta)}{c^2} - \alpha'^3\, K_3\,,\\
\frac{\dot f}{f} &= -\frac{\sqrt{2}\, f\, (\cos\tilde\delta - \sin\tilde\delta)}{c^2} + \frac{\sqrt{2}\, f\, \cos\tilde\delta}{a^2} - \frac{\sqrt{2}\, f\, \sin\tilde\delta}{b^2} - \alpha^3\, K_4\,,
\end{aligned} \tag{5.14}$$

where the K_i's are polynomial functions in a, b, c and f. (We have temporarily absorbed the constant c into α'^3 in the discussion of this example, to avoid confusion with the metric function c.) We have explicitly verified that these first-order equations satisfy the generalised higher-order second-order Einstein equations. Owing to the complexity of the expressions for the K_i's, we shall not present their general form, but give only a certain specific example.

Local solutions of the first-order equations for Spin(7) holonomy exist for all values of k and ℓ [12]. In general these have conical singularities, but in the special case $N(1,0)$, or its permutation-related cousins $N(0,1)$ or $N(1,-1)$, then the solution, first found in [13], is complete and non-singular. The solution is given by

$$\bar a = \sqrt{(r-1)(r+5)}\,, \qquad \bar b = (r+1)\,, \qquad \bar c = \sqrt{r^2-9}\,, \qquad \bar f = -\sqrt{\frac{9(r-3)(r+5)}{2(r+3)(r-1)}}\,, \tag{5.15}$$

where the coordinate r is related to t by $dt = h\, dr \equiv -\frac{3}{\sqrt{2}} f^{-1}\, dr$. Note that we use barred notation to denote the background variables. For this specific metric, we find that

$$\begin{aligned}
K_1 &= \frac{162(4r^8 - 13r^7 - 83r^6 - 409r^5 + 81r^4 - 1351r^3 - 3993r^2 - 39955r - 97641}{h\,(r-1)^8(r+3)^7}\,,\\
K_2 &= \frac{648(r+1)(r^6 + 6r^5 - 18r^4 - 112r^3 - 91r^2 + 58r - 5604)}{h\,(r-1)^7(r+3)^7}\,,\\
K_3 &= \frac{162(4r^8 + 77r^7 + 547r^6 + 2297r^5 + 7311r^4 + 19527r^3 + 34761r^2 + 69491r - 11135)}{h\,(r-1)^7(r+3)^8}\,,\\
K_4 &= \frac{2592(r+1)(r^2+2r-43)(3r^4+12r^3-170r^2-364r-1049)}{h\,(r-1)^8(r+3)^8}\,.
\end{aligned} \tag{5.16}$$

6. Deformation of Spin(7) compactifications of M-theory

In this section, we now consider analogous corrections to an initial $(\text{Minkowski})_3 \times K_8$ background in M-theory, which is related by dimensional reduction to type-IIA string theory at the one string-loop level. To begin, we give a general discussion of the known correction terms in the M-theory effective action.

6.1 Corrections to (Minkowski)$_3 \times K_8$ backgrounds

The corrections to the $D = 11$ bosonic lagrangian, which correspond to the lift of 1-loop corrections in the type-IIA string, take the form

$$\mathcal{L}_1 = -\frac{\beta}{1152}(\hat{Y} + 2\hat{Y}_2 + \cdots)\,\hat{*}\mathbb{1} + \beta\,(2\pi)^4\,\hat{A}_{(3)} \wedge \hat{X}_{(8)}\,, \tag{6.1}$$

where $\hat{X}_{(8)}$ is given by

$$\hat{X}_{(8)} = \frac{1}{192\,(2\pi)^4}\left[\mathrm{tr}\,\hat{\Theta}^4 - \frac{1}{4}(\mathrm{tr}\,\hat{\Theta}^2)^2\right], \tag{6.2}$$

and $\hat{Y}$ and $\hat{Y}_2$ are eleven-dimensional analogues of the ten-dimensional quantities Y and Y_2 described in section 2, but now with the summation index ranges extended to 11 rather than 10 values. In particular, $\hat{Y}_2$ is proportional to the covariant generalisation of the eight-dimensional Euler integrand,

$$\hat{Y}_2 = \frac{315}{2}\hat{R}^{[M_1 M_2}{}_{M_1 M_2} \cdots \hat{R}^{M_7 M_8]}{}_{M_7 M_8}\,. \tag{6.3}$$

The constant β now takes on the rôle played by α'^3 in string theory, and we shall work to order β in the subsequent discussion.

The ellipses in (6.1) represent terms that vanish by use of the leading-order field equations, and which therefore can be adjusted by choice of field variables. These changes of variable do not, of course, affect the physics, but they can be used to advantage in order to make the discussion more elegant. By adding a specific term of this type, we shall be able to ensure that the corrected equations of motion describing the modification to the Spin(7) holonomy internal space are the same as those that we found at tree-level in string theory. To achieve this, we shall take the bracketed volume term in (6.1) to be

$$\hat{W} = \hat{Y} + 2\hat{Y}_2 - \hat{R}\,\hat{Z}\,, \tag{6.4}$$

and so

$$\mathcal{L}_1 = -\frac{\beta}{1152}\,\hat{W}\,\hat{*}\mathbb{1} + (2\pi)^4\,\beta\,\hat{A}_{(3)} \wedge \hat{X}_{(8)}\,, \tag{6.5}$$

The additional $\hat{R}\,\hat{Z}$ term is introduced for convenience by a field rededinition of the metric, as in [2], to compensate for the absence of a dilaton in M-theory. It does not change the physics, but it renders the equations more elegant.

The variation $\delta \int \sqrt{-\hat{g}}\,\hat{Y}\,d^{11}x \equiv \int \sqrt{-\hat{g}}\,\hat{Y}_{MN}\,\delta\hat{g}^{MN}\,d^{11}x$ yields

$$\hat{Y}_{\mu\nu} = 0\,, \qquad \hat{Y}_{ij} = X_{ij}\,, \tag{6.6}$$

in the 3-dimensional spacetime and the internal 8-dimensional manifold respectively, after imposing the leading-order (Minkowski)$_3 \times M_8$ background conditions, where M_8 is a Spin(7) manifold. The tensor X_{ij} is given by (2.28). Varying $\hat{Y}' \equiv (\hat{Y} - \hat{R}\,\hat{Z})$ instead of $\hat{Y}$, we find

$$\hat{Y}'_{\mu\nu} = -g_{\mu\nu}\,\Box Z\,, \qquad \hat{Y}'_{ij} = X_{ij} + \nabla_i\nabla_j Z - g_{ij}\,\Box Z\,, \tag{6.7}$$

after imposing the (Minkowski)$_3 \times M_8$ background equations. The variation of the additional $D = 8$ Euler integrand term $2\hat{Y}_2\sqrt{-\hat{g}}$ yields a contribution $-\hat{g}_{\mu\nu}\,\hat{Y}_2$ in the 3 spacetime directions, and zero in the internal directions (since $\hat{Y}_2$ is topological in eight dimensions).

The variation of the full $\hat{W}$ term in the M-theory effective action therefore leads to the corrected Einstein equations

$$\hat{R}_{\mu\nu} - \frac{1}{2}\hat{R}\,\hat{g}_{\mu\nu} = -\frac{\beta}{1152}\,(\Box Z + Y_2)\,g_{\mu\nu}\,, \tag{6.8}$$
$$\hat{R}_{ij} - \frac{1}{2}\hat{R}\,\hat{g}_{ij} = \frac{\beta}{1152}\,(X_{ij} + \nabla_i\nabla_j\,Z - g_{ij}\,\Box Z)\,, \tag{6.9}$$

after imposing the (Minkowski)$_3 \times M_8$ structure in the β correction terms. We do not need to include the energy-momentum tensor of the 4-form here, since $\hat{F}_{(4)}$ is taken to vanish at leading order, and thus it itself will be of order β in the corrected solutions and so it would contribute only at order β^2 in the Einstein equations. For the same reason, we do not need to include the contribution to the Einstein equation that would come from varying the metrics in the $\hat{A}_3 \wedge \hat{X}_8$ term in (6.5), since it already carries a factor of β, and since the resulting $\hat{F}_4$ will also be small, of order β.

The corrected field equation for $\hat{F}_{(4)}$ is

$$d\hat{*}\hat{F}_{(4)} = \frac{1}{2}\hat{F}_{(4)} \wedge \hat{F}_{(4)} + (2\pi)^4\,\beta\,\hat{X}_{(8)}\,. \tag{6.10}$$

The 4-form and the eleven-dimensional metric will be required to have the 3-dimensional Poincaré invariance of the leading-order solution, which implies that we can write

$$d\hat{s}_{11}^2 = e^{2A}\,\eta_{\mu\nu}\,dx^\mu\,dx^\nu + e^{-A}\,ds_8^2\,, \tag{6.11}$$
$$\hat{F}_4 = d^3x \wedge df + G_{(4)}\,, \tag{6.12}$$

where A and f are functions only of the coordinates on M_8, and $G_{(4)}$ is a 4-form residing purely in the internal space.

6.2 Spin(7) non-compact solutions

The discussion that follows will be similar to one given in ref. [15]. Since we are working only to order β in this discussion, we can consider separately the contributions of the two terms in the field-strength ansatz (6.12). The former is obligatory, in the sense that the local equation of motion (6.10) forces f to become non-zero (and of order β). In contrast, the inclusion of the second term $G_{(4)}$ in (6.12) is optional if the "internal" space K_8 is non-compact; in particular it can be chosen to be zero. To proceed, we consider this case first, subsequently returning to consider the modifications needed for compact K_8.

The Ricci tensor of the metric (6.11) has non-vanishing coordinate-frame components given by

$$\hat{R}_{\mu\nu} = -e^{3A}\,\Box A\,\eta_{\mu\nu}\,, \tag{6.13}$$
$$\hat{R}_{ij} = R_{ij} + \frac{1}{2}\,\Box A\,g_{ij} - \frac{9}{2}\nabla_i A\,\nabla_j A\,, \tag{6.14}$$

where R_{ij} is the Ricci tensor of $ds_8^2 = g_{ij}\, dy^i\, dy^j$. Note that since we shall be working to order β, and since the leading-order background is $d\hat{s}_{11}^2 = \eta_{\mu\nu}\, dx^\mu\, dx^\nu + ds_8^2$ where ds_8^2 is Ricci-flat, we may neglect the terms quadratic in ∇A in the expression for $\hat{R}_{ij}$, since we shall have

$$A = 0 + \mathcal{O}(\beta)\,. \tag{6.15}$$

Similarly, exponential factors of e^A that multiply quantities that are already of order β may be replaced by 1. We shall drop all such higher-order terms in what follows. In particular, we may write (6.14) simply as

$$\begin{aligned} \hat{R}_{\mu\nu} &= -\Box A\, \eta_{\mu\nu}\,, & (6.16)\\ \hat{R}_{ij} &= R_{ij} + \frac{1}{2}\Box A\, g_{ij}\,. & (6.17) \end{aligned}$$

From (6.16) and (6.17) we find $\hat{R} = R + \Box A$, and hence by substituting (6.17) into (6.9) we find

$$R_{ij} - \frac{1}{2} R\, g_{ij} = \frac{\beta}{1152}\,(X_{ij} + \nabla_i \nabla_j\, Z - g_{ij}\, \Box Z)\,. \tag{6.18}$$

Taking the trace gives $R = (\beta/576)\,\Box Z$, and hence (6.18) yields

$$R_{ij} = \frac{\beta}{1152}\,(X_{ij} + \nabla_i \nabla_j\, Z)\,. \tag{6.19}$$

From (6.8) we then find

$$\Box A = \frac{\beta}{1728}\, Y_2\,. \tag{6.20}$$

Equations (6.19) and (6.20) comprise the final expressions that follow from the corrected Einstein equations (6.8) and (6.9). It is important to note that all terms involving $\Box Z$ have cancelled.[7] This depends, in particular, on the fact that $X_{ij}\, g^{ij} = \Box Z$, which was shown for a Spin(7) background in (2.29). Note that the correction to the Ricci-flatness of the leading-order Spin(7) manifold, described by (6.19), is identical to the corrected equation (1.11) that we obtained at tree level in string theory.

Again working to order β, substitution of the ansatz (6.12) into the corrected 4-form equation (6.10) yields $d{*}df = \beta\, (2\pi)^4\, X_8$, or, after dualization

$$\Box\, f = \beta\, (2\pi)^4\, {*}X_8\,. \tag{6.21}$$

If the internal space M_8 admits a nowhere-vanishing spinor, as is always the case on a space of special holonomy, there is a topological relation between the Euler class E_8 and the combination of P_2 and P_1^2 Pontryagin classes that arises in X_8 [19, 20]. This translates into the statement that

$$Y_2 = 576 (2\pi)^4\, {*}X_8\,. \tag{6.22}$$

[7] The analogous cancellation did not occur in the discussion presented in [15] for deformations of eight-dimensional Ricci-flat Kähler backgrounds, but this is simply because a different choice of field variables was used there. Earlier papers, including [14, 16, 18], did not include the contributions from the volume terms $\hat{Y}$ and $\hat{Y}_2$ in (6.5) at all, and so the "M2-brane like" metric ansatz (6.11) that was made in those papers would have been in conflict with the Einstein equations in the spacetime directions at order β (see (6.13), (6.14), (6.16) and (6.17)).

Comparing (6.20) and (6.21), this implies (for non-singular solutions without δ-function sources) that we must have

$$f = 3A\,. \tag{6.23}$$

As we shall now show, this is in fact precisely the condition that is needed in order to ensure that the deformed solution will still be supersymmetric.

6.3 Supersymmetry of the deformed (Minkowski)$_3 \times K_8$ background

The classical gravitino transformation rule in eleven-dimensional supergravity takes the form

$$\delta\hat\psi_M = \hat\nabla_M\,\hat\epsilon - \frac{1}{288}\hat F_{N_1\cdots N_4}\,\hat\Gamma_M{}^{N_1\cdots N_4}\,\hat\epsilon + \frac{1}{36}\hat F_{MN_1\cdots N_3}\,\hat\Gamma^{N_1\cdots N_3}\,\hat\epsilon\,. \tag{6.24}$$

We shall use the following $11 = 3+8$ decomposition of the eleven-dimensional Dirac matrices $\hat\Gamma_M$:

$$\hat\Gamma_\mu = \gamma_\mu\otimes\Gamma_9\,,\qquad \hat\Gamma_i = \mathbb{1}\otimes\Gamma_i\,, \tag{6.25}$$

where Γ_9 is the chirality operator in the eight-dimensional internal space. To the order β that we are working, it suffices to retain the contributions from the field strength $\hat F_{(4)}$ and the metric warp factor A only up to linear order. From (6.11), we therefore find that in the natural choice of spinor frame, the covariant derivative $\hat\nabla_M$ in the spacetime and internal directions is given by

$$\hat\nabla_\mu = \partial_\mu\otimes\mathbb{1} + \frac12\partial_i A\,\gamma_\mu\otimes\Gamma_9\Gamma^i\,,\qquad \hat\nabla_i = \mathbb{1}\otimes\nabla_i - \frac14\partial_j A\,\mathbb{1}\otimes\Gamma_i{}^j\,. \tag{6.26}$$

Including the contribution of the 4-form, which is given by (6.12), we therefore have the supersymmetry transformation $\delta\hat\psi_M = \hat D_M\,\hat\epsilon$, where

$$\begin{aligned}\hat D_\mu &= \partial_\mu - \frac12\partial_i A\,\gamma_\mu\otimes\Gamma^i\,\Gamma_9 - \frac16\partial_i f\,\gamma_\mu\otimes\Gamma^i\,,\\ \hat D_i &= \mathbb{1}\otimes\nabla_i - \frac14\partial_j A\,\mathbb{1}\times\Gamma_i{}^j - \frac{1}{12}\partial_j f\,\mathbb{1}\otimes\Gamma_i{}^j\,\Gamma_9 + \frac16\partial_i f\,\mathbb{1}\otimes\Gamma_9 + \mathbb{1}\otimes Q_i\,,\end{aligned} \tag{6.27}$$

and Q_i is the correction to the supersymmetry transformation discussed in section 3. It is straightforward to verify that the Killing spinor condition $\hat D_M\,\hat\epsilon = 0$ is satisfied if we write

$$\hat\epsilon = e^{\frac12 A}\,\epsilon\otimes\eta\,, \tag{6.28}$$

where ϵ is a constant spinor in the 3-dimensional Minkowski spacetime, and η is a chiral spinor in the internal 8-dimensional space, $\Gamma_9\,\eta = -\eta$, which satisfies the usual modified covariant-constancy condition

$$\nabla_i\,\eta + Q_i\,\eta = 0 \tag{6.29}$$

that we discussed previously in the context of tree-level string corrections.

Note that the additional ingredients in the current M-theory discussion, in comparison to our previous tree-level string discussion, are associated with the warp factor appearing in the metric (6.11), and the field strength (6.12) that is forced to be non-zero because of the $\hat A_3\wedge\hat X_8$ term in the effective action. These two contributions in the supercovariant derivatives (6.27) cancel against each other, by virtue of (6.23), in exactly the same way as one finds in a standard M2-brane solution [21] of eleven-dimensional supergravity.

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6.4 Compact K_8

When the internal manifold K_8 is non-compact then the inclusion of the term $G_{(4)}$ in the field-strength ansatz (6.12) is optional. However, when K_8 is a compact manifold of non-zero Euler number there is an additional topological condition that follows by integrating (6.10), namely [16]

$$\int_{K_8} G_{(4)} \wedge G_{(4)} = \frac{(2\pi)^4\,\beta}{12}\,\chi\,, \tag{6.30}$$

where χ is the Euler number of K_8. Under these circumstances, the inclusion of the term $G_{(4)}$ in (6.12) becomes obligatory; clearly we must take

$$G_{(4)} = \sqrt{\beta}\,\omega_{(4)} \tag{6.31}$$

where $\omega_{(4)}$ is a closed 4-form on K_8 that we take to be β-independent. It must also be co-closed in order to avoid an order $\sqrt{\beta}$ correction in (6.10). There is also a potential order $\sqrt{\beta}$ correction to the supercovariant derivatives (6.27), namely

$$\begin{aligned}
\hat{D}_\mu &\longrightarrow \hat{D}_\mu - \sqrt{\beta}\frac{1}{288}\,\omega_{j_1\cdots j_4}\,\gamma_\mu \otimes \Gamma^{j_1\cdots j_4}\,,\\
\hat{D}_i &\longrightarrow \hat{D}_i - \sqrt{\beta}\frac{1}{288}\,\mathbb{1} \otimes \big(\omega_{j_1\cdots j_4}\,\Gamma_i{}^{j_1\cdots j_4} - 8\omega_{ij_1\cdots j_3}\,\Gamma^{j_1\cdots j_3}\big)\,.
\end{aligned} \tag{6.32}$$

The $\sqrt{\beta}$ corrections cancel if

$$\omega_{ij_1\cdots j_3}\,\Gamma^{j_1\cdots j_3}\,\eta = 0 \tag{6.33}$$

is satisfied. This can be viewed as a supersymmetry-preservation condition on the internal 4-form $\omega_{(4)}$. It implies that $\omega_{(4)}$ must be self-dual [14, 16, 17] (which is the same sense of duality as for the calibrating 4-form $c_{(4)}$ given by (2.9)), and hence that it must be closed as well as co-closed. In other words, ω_4 must be a self-dual harmonic 4-form. Note, however, that $c_{(4)}$ is not a suitable candidate for $\omega_{(4)}$ because if we left-multiply (6.33) by $\bar{\eta}\Gamma^i$ we get $\omega_{ijk\ell}\,c^{ijk\ell} = 0$, and this is not satisfied by $\omega_4 = c_4$. What this shows is that ω_4 must be a self-dual harmonic 4-form that is *orthogonal* to $c_{(4)}$.

It is useful at this point to look at the decomposition of the SO(8) tangent-space representations of 4-forms under the Spin(7) holonomy group. We have

$$\mathbf{35}_+ \longrightarrow \mathbf{1} + \mathbf{7} + \mathbf{27}\,,\qquad \mathbf{35}_- \longrightarrow \mathbf{35}\,, \tag{6.34}$$

for self-dual and anti-self-dual 4-forms respectively. Since this decomposition is made with respect to the invariant calibrating 4-form, which defines the Spin(7) embedding in SO(8), it follows that the decomposition commutes with covariant differentiation. This allows a refinement of the cohomology for self-dual 4-forms, in which we may write [24]

$$H^4_+(K_8,\mathbb{R}) = H^4_1(K_8,\mathbb{R}) + H^4_7(K_8,\mathbb{R}) + H^4_{27}(K_8,\mathbb{R})\,. \tag{6.35}$$

Correspondingly, we have for the Betti numbers $b_4 = b_4^+ + b_4^-$, with $b_4^+ = b_4^{(1)} + b_4^{(7)} + b_4^{(27)}$. It is shown in [24] that for any compact Spin(7) manifold, $b_4^{(7)} = 0$, and $b_4^{(1)} = 1$. This last identity corresponds to the fact that the calibrating 4-form is the unique Spin(7)-invariant self-dual harmonic form. Thus we have that

$$b_4^+ = 1 + b_4^{(27)}\,, \tag{6.36}$$

and so *any* self-dual harmonic 4-form other than the calibrating 4-form can provide a solution that satisfies the supersymmetry condition (6.33).[8]

The fact that $\omega_{(4)}$ is closed takes care of any order $\sqrt{\beta}$ terms in (6.10), but we must now take into account the order β contribution from the $\hat{F}_4 \wedge \hat{F}_4$ term. This has the effect of modifying (6.21) to

$$\Box\, f = \beta\, [(2\pi)^4 *X_8 + \frac{1}{48}|\omega_{(4)}|^2]\,, \tag{6.37}$$

where we have used the self-duality of $\omega_{(4)}$ to write the dual of $\omega_{(4)} \wedge \omega_{(4)}$ as $\frac{1}{24}|\omega_{(4)}|^2$. There is a similar order β correction to the stress-tensor for $\hat{F}_4$ (which we were previously able to set to zero). This modifies the source for the Einstein equations on K_8, but the only effect of this is a modification of the source term of the Poisson equation (6.20), which becomes

$$\Box A = \beta \left[\frac{1}{1728}\, Y_2 + \frac{1}{144}\, |\omega_{(4)}|^2\right]. \tag{6.38}$$

Fortunately, the consistency of (6.37) and (6.38) is again assured because of (6.22), and again we find that $f = 3A$, just as we found for a non-compact K_8 with vanishing G_4. As we saw in the non-compact case, the equality $f = 3A$ is crucial for the supersymmetry of the deformed background. Note that this is also the relation found in ref. [25] from direct consideration of supersymmetry in the R^4 — Chern Simons system.

To summarise, we have shown that if $G_{(4)}$ is taken to be proportional to *any* self-dual harmonic 4-form other than the calibrating 4-form (i.e. any self-dual harmonic 4-form in the **27** of Spin(7)), the local equations of motion and the supersymmetry conditions are still satisfied by the deformed Spin(7) holonomy background, up to order β. Furthermore, one can always satisfy the global topological constraint (6.30), by normalising the harmonic self-dual 4-form appropriately, namely so that

$$\int_{K_8} |G_{(4)}|^2 = 2(2\pi)^4\, \beta\, \chi\,, \tag{6.39}$$

In [24], many examples of compact manifolds with Spin(7) holonomy are constructed, typically with large values of the Betti number b_4^+ of self-dual harmonic 4-forms. In fact from (6.36), we see that whenever b_4^+ is greater than 1, there will exist suitable self-dual harmonic 4-forms that allow the global condition (6.30) to be satisfied.

It is also worth noting that if K_8 is non-compact, in which case the inclusion of a non-vanishing $G_{(4)}$ is optional rather than obligatory, explicit constructions of self-dual harmonic forms that satisfy the supersymmetry condition (6.33) are known [11, 26, 27].

[8] In fact, a more detailed investigation of (6.33) reveals that it already selects precisely self-dual 4-forms in the **27** representation of Spin(7), quite independently of the above discussion of the refined cohomology.

7. Deformation of SU(5) holonomy solutions of M-theory

We now turn to compactifications of M-theory on ten-dimensional manifolds K_{10} which, at leading order, are Ricci-flat and Kähler. It should be emphasised that such backgrounds probe aspects of M-theory that go beyond anything that can be directly deduced from light-cone string-theory computations, which, in practice, have provided most of the concrete information about the structure of M-theory. In fact, SU(5) holonomy backgrounds cannot be discussed at all in perturbative string theory, since there are only nine euclidean-signature dimensions. Thus not only do SU(5) holonomy backgrounds go beyond what can be learned from light-cone string-theory calculations, they go beyond perturbative string theory itself, and are intrinsic to M-theory. Nevertheless, it has been argued that the information learned from light-cone string calculations, and elsewhere, can be extrapolated to genuinely eleven-dimensional results about the structure of M-theory. It is therefore of interest to see what happens if one tries to "push the envelope" and apply these eleven-dimensional results to SU(5) holonomy backgrounds.

7.1 Leading-order preliminaries

To set up our discussion of corrections to SU(5)-holonomy compactifications of M-theory, we will begin with a brief discussion of the leading-order SU(5)-holonomy compactifications of 11-dimensional supergravity. The (undeformed) solutions of interest have vanishing fermions, vanishing 4-form field strength F, and a metric of the form

$$ds^2 \equiv g_{MN} dx^M dx^N = -dt^2 + g_{ij} dx^i dx^j \tag{7.1}$$

where the 10-metric g_{ij} on K_{10} has SU(5) holonomy. The 11D Dirac matrices can be taken to be

$$\hat{\Gamma}_0 = i\gamma_{11}\,, \qquad \hat{\Gamma}_i = \gamma_i \tag{7.2}$$

where γ_i are the SO(10) Dirac matrices, and γ_{11} is the chirality operator on SO(10) spinors,

$$\gamma_{11} = i\gamma_1\gamma_2\cdots\gamma_{10}\,. \tag{7.3}$$

We will assume (in accordance with the usual custom) that the 11D Dirac matrices $\hat{\Gamma}_M$ are hermitian, in which case the SO(10) Dirac matrices γ_i are hermitian.

The supersymmetry-preservation condition for solutions of 11D supergravity is the vanishing of the supersymmetry variation of the gravitino. For purely gravitational backgrounds this reduces to

$$\hat{D}_M \hat{\epsilon} = 0 \tag{7.4}$$

where D_M is the covariant derivative on spinors and $\hat{\epsilon}$ is a Majorana spinor; i.e., it satisfies

$$\hat{\epsilon}^\dagger = \hat{\epsilon}^T \hat{C} \hat{\Gamma}_0 \tag{7.5}$$

where $\hat{C}$ is the antisymmetric SO(1, 10) charge conjugation matrix. For compactifications on K_{10}, the condition (7.4) reduces to the equation

$$D_i \hat{\epsilon} = 0, \tag{7.6}$$

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where $\hat{\epsilon}$ is now a time-independent SO(10) spinor on K_{10} and D_i is the covariant derivative on such spinors. The 11D Majorana condition (7.5) becomes

$$\hat{\epsilon}^* = C\hat{\epsilon}\,, \qquad C = \hat{C}\hat{\Gamma}_0 \tag{7.7}$$

where C is the real symmetric SO(10) charge conjugation matrix, with the property that

$$C\gamma_i C^{-1} = \gamma_i^T\,. \tag{7.8}$$

Equivalently, since the matrices γ_i are hermitian,

$$C\gamma_i C^{-1} = \gamma_i^*\,. \tag{7.9}$$

One also has $C^2 = 1$, so one may choose a basis such that $C = 1$, in which case the matrices γ_i are real, as are Majorana spinors. However, in this basis γ_{11} is pure imaginary, so the Majorana condition is not compatible with a chirality condition. This result is, of course, basis independent, so a 'minimal' SO(10) spinor is *either* Majorana *or* complex chiral.

For some purposes it is simpler to work with complex chiral SO(10) spinors. In particular, a 10-manifold of SU(5) holonomy admits one covariantly constant complex chiral spinor, as follows from the decomposition

$$\mathbf{16} = \mathbf{10} \oplus \mathbf{5} \oplus \mathbf{1} \tag{7.10}$$

of the spinor irrep of Spin(10) into irreps of SU(5). Let η be this one chiral spinor; we choose conventions such that the chirality condition is

$$\gamma_{11}\eta = -\eta\,. \tag{7.11}$$

Note that the charge conjugate spinor

$$\eta^c := C^{-1}\eta^* \tag{7.12}$$

satisfies the anti-chirality condition $\gamma_{11}\eta^c = \eta^c$ as a consequence of the identity (for hermitian Dirac matrices)

$$C\gamma_{11}C^{-1} = -\gamma_{11}^*\,. \tag{7.13}$$

Moreover, as a consequence of the identity

$$C\gamma_{ij}C^{-1} = \gamma_{ij}^*\,, \tag{7.14}$$

the spinor η^c is covariantly constant if η is covariantly constant. An alternative way to see this is to note that the covariant derivative is real in a real basis for the Dirac matrices, so that in such a basis the real and imaginary parts of a covariantly constant complex spinor are covariantly constant Majorana spinors. In particular, the existence of one covariantly constant chiral spinor η implies the existence of two linearly independent covariantly constant Majorana spinors, defined by

$$\epsilon_1 = \frac{1}{2}\left(\eta + \eta^c\right), \qquad \epsilon_2 = -\frac{i}{2}\left(\eta - \eta^c\right). \tag{7.15}$$

Using $C^2 = 1$, it is easily verified that these spinors *are* Majorana. They are covariantly constant because $D_i \eta = 0$ implies $D_i \eta^c = 0$. Note that

$$\eta = \epsilon_1 + i\epsilon_2 \tag{7.16}$$

which is the decomposition of a complex spinor into two Majorana spinors; in a real basis, for which $C = 1$, the Majorana spinors ϵ_1 and ϵ_2 are just the real and imaginary parts of the complex spinor η. If η is a chiral spinor, satisfying (7.11) then

$$\epsilon_2 = i\gamma_{11}\epsilon_1 \,. \tag{7.17}$$

However, the Majorana spinors ϵ_1 and ϵ_2 are still linearly independent over the reals because the linear combination $(a + ib\gamma_{11})\epsilon_1$ vanishes for real numbers a, b, and non-zero ϵ_1, if and only if $a = b = 0$.

We have thus shown that, when K_{10} is a manifold of SU(5) holonomy, there are two linearly-independent Majorana spinor solutions of (7.6), and hence of the supersymmetry preservation condition (7.4), and that this statement is equivalent to the statement that K_{10} admits a single complex chiral Killing spinor.

For future use we also note that

$$\gamma_{\hat{j}}\, \eta = \mathrm{i}\, \gamma_j\, \eta \,, \qquad (\gamma_{ij} + \gamma_{\hat{i}\hat{j}})\, \eta = 2\mathrm{i}\, J_{ij}\, \eta \,, \tag{7.18}$$

where J_{ij} is the Kähler form, and we are using the "hat" notation of [8], defined in (4.6). Other useful properties following from these are

$$\bar\eta\, \gamma_{ij}\, \eta = \mathrm{i}\, J_{ij} \,, \qquad \bar\eta\, \gamma_{ijk\ell}\, \eta = -J_{ij}\, J_{k\ell} - J_{ik}\, J_{\ell j} - J_{i\ell}\, J_{jk} \,. \tag{7.19}$$

7.2 Corrections to (Minkowski)$_1 \times K_{10}$ backgrounds

The relevant $\mathcal{O}(\beta)$ corrections to the equations of motion again follow from (6.5). The contributions from the eight-dimensional Euler integrand term $\hat{Y}_2 \sqrt{-\hat{g}}$ can be determined by varying the explicit metrics needed to write $\sqrt{-\hat{g}}$ times the right-hand side of (6.3) in terms of canonical Riemann tensors $\hat{R}^M{}_{NPQ}$ with one index up and three down. (One does not need to vary the metrics from which $\hat{R}^M{}_{NPQ}$ is constructed, since these variation terms will be of the form of a total derivative, and hence will not contribute in the equations of motion.[9]) Thus defining $\delta \int \hat{Y}_2 \sqrt{-\hat{g}} = \int \sqrt{-\hat{g}}\, \hat{E}_{MN}\, \delta \hat{g}^{MN}$, one finds (see, for example, [22])

$$\hat{E}_M{}^N = -\frac{9!}{2^9}\, \delta^{NN_1\cdots N_8}_{MM_1\cdots M_8}\, \hat{R}^{M_1M_2}{}_{N_1N_2} \cdots \hat{R}^{M_7M_8}{}_{N_7N_8} \,, \tag{7.20}$$

where the Kronecker deltas are of unit strength ($\delta^{N_1\cdots N_n}_{M_1\cdots M_n} \omega_{N_1\cdots N_n} = \omega_{M_1\cdots M_n}$ for any anti-symmetric tensor $\omega_{M_1\cdots M_n}$).

[9]In the same way, the terms from the metrics in $R_{MN} = R^P{}_{MPN}$ do not contribute when one varies the two-dimensional Euler integrand $g^{MN}\, R_{MN} \sqrt{-g}$ (the Einstein-Hilbert action) to obtain the Einstein tensor.

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The eleven-dimensional Einstein equations, with their $\mathcal{O}(\beta)$ corrections, are given by

$$
\begin{aligned}
\hat{R}_{00} - \frac{1}{2}\hat{R}\,\hat{g}_{00} &= -\frac{\beta}{1152}\,\Box Z\, g_{00} + \frac{\beta}{576}\,\hat{E}_{00}\,, && (7.21)\\
\hat{R}_{ij} - \frac{1}{2}\hat{R}\,\hat{g}_{ij} &= \frac{\beta}{1152}\,(X_{ij} + \nabla_i\nabla_j\, Z - g_{ij}\,\Box Z) + \frac{\beta}{576}\,\hat{E}_{ij}\,, && (7.22)
\end{aligned}
$$

where

$$
Z = R_{ijk\ell}\, R^{k\ell mn}\, R_{mn}{}^{ij} - 2R_{ikj\ell}\, R^{km\ell n}\, R_m{}^i{}_n{}^j\,, \tag{7.23}
$$

after imposing the $(\text{Minkowski})_1 \times K_{10}$ Ricci-flat Kähler background conditions in the correction terms on the right-hand sides. Note that we shall have

$$
\begin{aligned}
\hat{E}_{00} &= \frac{1}{2}Y_2\,,\\
\hat{E}_i^{\ j} &= E_i^{\ j} \equiv -\frac{9!}{2^9}\,\delta^{jj_1\cdots j_8}_{ii_1\cdots i_8}\, R^{i_1 i_2}{}_{j_1 j_2}\cdots R^{i_7 i_8}{}_{j_7 j_8}\,, && (7.24)
\end{aligned}
$$

in the $(\text{Minkowski})_1 \times K_{10}$ background. The new feature that we encounter here, in comparison to the $(\text{Minkowski})_3 \times K_8$ backgrounds described by (6.8) and (6.9), is that in (7.22) we have the non-zero contribution $\hat{E}_{ij}$ coming from the variation of the eight-dimensional Euler integrand. It is manifest from its form, given in (7.24), that this would vanish in an 8-dimensional curved background, owing to the antisymmetrisation over 9 indices.

As in the case of $(\text{Minkowski})_3 \times K_8$ backgrounds, we expect that the effect of the order β corrections to the $(\text{Minkowski})_1 \times K_{10}$ background will be to introduce a warp factor in the eleven-dimensional metric, as well as causing the originally-vanishing 4-form to become non-zero. For the metric, we therefore write

$$
d\hat{s}_{11}^2 = -e^{2A}\, dt^2 + e^{-\frac{1}{4}A}\, ds_{10}^2\,, \tag{7.25}
$$

where the function A in the warp factor depends only on the coordinates of K_{10}. The relative powers of the warp factor in the two terms in (7.25) are motivated by the expectation of a "0-brane" structure in the deformed solution. At the linearised level, which suffices for our purposes since we are perturbing around the original background with $A = 0$ and K_{10} Ricci-flat and Kähler, we find that the non-vanishing Riemann tensor components for the metric (7.25) are given by

$$
\begin{aligned}
\hat{R}_{0i0j} &= \nabla_i\nabla_j A\,, && (7.26)\\
\hat{R}_{ijk\ell} &= R_{ijk\ell} - \frac{1}{8}(g_{i\ell}\,\nabla_j\nabla_k A - g_{ik}\,\nabla_j\nabla_\ell A + g_{jk}\,\nabla_i\nabla_\ell A - g_{j\ell}\,\nabla_i\nabla_k A)\,, && (7.27)
\end{aligned}
$$

and the non-vanishing components of the Ricci tensor are given by

$$
\hat{R}_{00} = \Box A\,, \qquad \hat{R}_{ij} = R_{ij} + \frac{1}{8}g_{ij}\,\Box A\,. \tag{7.28}
$$

Taking the eleven-dimensional trace gives $\hat{R} = R + \frac{1}{4}\Box A$, and substituting this into (7.22) and tracing leads to

$$
R = \frac{\beta}{576}\,\Box Z - \frac{\beta}{2304}\, E_i^{\ i}\,. \tag{7.29}
$$

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Equation (7.22) then gives

$$R_{ij} = \frac{\beta}{1152}\Big(X_{ij} + \nabla_i\nabla_j\, Z + 2E_{ij} - \frac{1}{4}E_k{}^k\, g_{ij}\Big)\,. \tag{7.30}$$

Note that X_{ij}, coming from the variation of the "string tree-level" term $\hat{Y}$, is given by $X_{ij} = \nabla_{\hat{i}}\nabla_{\hat{j}}\, Z \equiv J_i{}^k\, J_j{}^\ell\, \nabla_k\nabla_\ell\, Z$, as usual in a Kähler background. Note also that from (7.20) we shall have

$$E_k{}^k = -Y_2\,. \tag{7.31}$$

The remaining content of the Einstein equations is contained in (7.21). From (7.29) and (7.31), we find that this implies

$$\Box A = \frac{\beta}{1728}\, Y_2\,. \tag{7.32}$$

After using (7.31), equation (7.30) can be written as

$$R_{ij} = \frac{\beta}{1152}\Big(\nabla_{\hat{i}}\nabla_{\hat{j}}\, Z + \nabla_i\nabla_j\, Z + 2E_{ij} + \frac{1}{4}Y_2\, g_{ij}\Big)\,. \tag{7.33}$$

Equations (7.32) and (7.33) determine the warp factor and the Ricci tensor of the corrected ten-dimensional Kähler metric, respectively. The field equation (6.10) will govern the structure of the non-vanishing 4-form that is required at order β. In order to maintain the 1-dimensional "Poincaré symmetry" of the original uncorrected background, it must be that

$$\hat{F}_{(4)} = G_{(3)} \wedge dt + G_{(4)}\,, \tag{7.34}$$

where $G_{(3)}$ and $G_{(4)}$ are 3-form and 4-form fields on K_{10}. We may, to begin with, assume that $G_{(4)} = 0$. The 4-form equation of motion (6.10) then implies, up to order β, that we shall have

$$d{*}G_{(3)} = (2\pi)^4\, \beta\, X_8\,, \tag{7.35}$$

where the unhatted $*$ denotes Hodge dualization in K_{10}.

Since the integrability condition obtained by taking the exterior derivative of this equation is trivially satisfied, we are guaranteed to be able to find a local solution of (7.35). However, integration over any 8-cycle C_8 of K_{10} leads to

$$\int_{C_8} X_8 = 0\,, \tag{7.36}$$

which must be satisfied for *all* 8-cycles C_8. This is a topological constraint on K_{10} that will not in general be satisfied unless $H_8(K_{10})$ is trivial. As Poincaré duality implies that $H_8 \cong H_2$ for any compact 10-manifold, and as H_2 is necessarily non-trivial for any Kähler manifold, the topological constraint is not satisfied by any compact Kähler 10-manifold; in other words, it is not satisfied by any compact manifold K_{10} of SU(5)-holonomy. What this means is that it is inconsistent to set $G_{(4)}$ to zero (as we have been doing) when K_{10} is compact. In principle, we could attempt to take this into account as we did in the Spin(7) case by allowing for a non-zero $G_{(4)}$ of order $\sqrt{\beta}$. However, the implications for supersymmetry are much less straightforward than they were for Spin(7) compactifications, so we shall not attempt an analysis along these lines here. Instead, we shall simply restrict discussion to the class of SU(5)-holonomy manifolds K_{10} for which H_8 is trivial. This implies that K_{10} is non-compact, so we are restricted to a special class of SU(5)-holonomy 'non-compactifications'.

With this restriction understood, the results above show that we can obtain an M-theory corrected solution, at order β, to the original $(\text{Minkowski})_1 \times K_{10}$ vacuum of $D = 11$ supergravity. The corrected metric is of the form of a warped product (7.25), with the warp factor given by (7.32), and the Ricci tensor of K_{10} given by (7.33). In the next subsection, we shall analyse the question of whether this M-theory corrected solution preserves the supersymmetry of the original Ricci-flat Kähler solution of $D = 11$ supergravity.

7.3 Supersymmetry of the deformed $(\text{Minkowski})_1 \times K_{10}$ backgrounds

We have seen in the previous subsection that the Ricci tensor of the originally Ricci-flat ten-dimensional Kähler space K_{10} suffers a more substantial deformation than has been seen hitherto for spaces K_n of special holonomy with $n \leq 8$, on account of the E_{ij} and $Y_2\, g_{ij}$ terms in (7.33) that come from the variation of the Euler integrand $\hat{Y}_2$.

It is of interest now to study the supersymmetry of the corrected $(\text{Minkowski})_1 \times K_{10}$ backgrounds. Here, we are on somewhat less solid ground. Although there has been a lot of work on the detailed structure of the higher-order corrections to supergravities in ten and eleven dimensions (see, for example, [23]), there are not, as far as we are aware, complete and explicit results for the corrections to the supersymmetry transformation rules at order α'^3 (or order β). The only explicit results are those introduced in [4] in the context of corrections to six-dimensional Calabi-Yau compactifications, their extension in [2] to G_2-holonomy compactifications, and their extension in the present paper to Spin(7)-holonomy compactifications. These corrections were deduced on the basis of *requiring* that the unbroken supersymmetry of the leading-order background should persist in the face of the α'^3 corrections.[10] Remarkably, the same riemannian expression (3.4) that was first proposed in [4] in the six-dimensional Calabi-Yau context has turned out to be sufficient to achieve a preservation of supersymmetry for the G_2 holonomy and Spin(7) holonomy backgrounds.

For an SU(5)-holonomy supergravity solution of 11D supergravity, we would again expect the M-theory correction to the gravitino transformation rule to lead to a modified covariant derivative $(\nabla_i + Q_i)$, where Q_i is of order β. If we assume that Q_i takes the same purely riemannian form[11] as in (3.4) then, using properties of SU(5) holonomy manifolds,

[10] This might seem somewhat circular as an argument for demonstrating that supersymmetry is preserved in the corrected special-holonomy backgrounds. However, the fact that one is able at all to find a candidate fully-riemannian correction to the gravitino transformation rule that is consistent with the preservation of supersymmetry of the corrected backgrounds is already quite remarkable. And since no other explicit results for the gravitino transformation rules have been obtained by direct calculation in the intervening 18 years since [4] appeared, we are forced, *faute de mieux*, to make do with this at present.

[11] Note that with the correction (3.4) the modified Killing spinor operator $(\nabla_i + Q_i)$ retains the same reality properties as at the classical level, so the equivalence between a pair of Majorana spinors and a complex chiral spinor as explained in subsection 7.1 persists in the presence of the corrections.

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one can show that

$$Q_i = \frac{\mathrm{i}\,\beta}{2304}\,\nabla_{\hat{i}} Z\,, \tag{7.37}$$

where Z is given by (7.23). There is no *a priori* reason why this assumption should be correct; there could be further terms whose presence would not be probed if one looked only at (Minkowski)$_3 \times K_8$ backgrounds, but which would be relevant to (Minkowski)$_1 \times K_{10}$ backgrounds. However, we shall show that this assumption nonetheless leads to the conclusion that supersymmetry of the corrected SU(5) holonomy backgounds is maintained, despite the loss of SU(5) holonomy. This is *a posteriori* evidence that the assumption is correct since one would hardly expect this conclusion to follow from an incorrect assumption, irrespective of whether supersymmetry is in fact preserved.

We begin by considering the integrability condition for the existence of a Killing spinor that satisfies $\hat{D}_M \hat{\epsilon} = 0$, obtained from the commutator of supercovariant derivatives. Since we are working only to linear order in β, and since the field strength $\hat{F}_{(4)}$ vanishes at zeroth order, becoming non-vanishing only at order β, we can omit terms quadratic in $\hat{F}_{(4)}$ in our discussion. We shall also suppress for now the $\mathcal{O}(\beta)$ Q_i correction to the supercovariant derivative; in other words, for now we shall just consider the "classical" terms in the integrability condition of $D = 11$ supergravity, with the added simplification of omitting the terms quadratic in $\hat{F}_{(4)}$. The contribution from Q_i will be included later, when we present our results. We therefore have for now that

$$[\hat{D}_M, \hat{D}_N]_0 = \frac{1}{4}\hat{R}_{MNPQ}\,\hat{\Gamma}^{PQ} + \frac{1}{144}\hat{\Gamma}_{[M}{}^{P_1\cdots P_4}\,\hat{\nabla}_{N]}\,\hat{F}_{P_1\cdots P_4} + \frac{1}{18}\,\hat{\nabla}_{[M}\,\hat{F}_{N]P_1P_2P_3}\,\hat{\Gamma}^{P_1P_2P_3}\,, \tag{7.38}$$

where the subscript "0" on the commutator indicates the omission of the Q_i correction term.

It is helpful to analyse the integrability conditions in stages. First, we may note that upon left-multiplication and contraction with $\hat{\Gamma}^N$, one obtains from $\hat{\Gamma}^N\,[\hat{D}_M, \hat{D}_N]\,\hat{\epsilon} = 0$ a system of field equations that can be compared with those already derived from the variation of the action. Thus if a Killing spinor $\hat{\epsilon}$ exists, one should find consistency between the already-established bosonic equations of motion, and those that follow from $\hat{\Gamma}^N\,[\hat{D}_M, \hat{D}_N]\,\hat{\epsilon} = 0$. Establishing this consistency does not of itself prove that a Killing spinor $\hat{\epsilon}$ exists (and thus that the deformed solution is supersymmetric), since the left-multiplication of the integrability condition by $\hat{\Gamma}^N$ projects into a subset of the full content of $[\hat{D}_M, \hat{D}_N]\,\hat{\epsilon} = 0$, but it already provides a non-trivial check.

It is easy to see from (7.38) that we shall have

$$\hat{\Gamma}^N\,[\hat{D}_M, \hat{D}_N]_0 = -\frac{1}{2}\hat{R}_{MN}\,\hat{\Gamma}^N - \frac{1}{72}\hat{\Gamma}_M{}^{N_1\cdots N_4}\,\hat{\nabla}^{N_1}\,\hat{F}_{N_1\cdots N_4} + \frac{1}{12}\hat{\nabla}^N\,\hat{F}_{NMPQ}\,\hat{\Gamma}^{PQ}\,. \tag{7.39}$$

The field equation (6.10) implies that

$$\hat{\nabla}_M\,\hat{F}^{MN_1N_2N_3} = \alpha\,\hat{\epsilon}^{N_1N_2N_3P_1\cdots P_8}\,\hat{X}_{P_1\cdots P_8}\,, \tag{7.40}$$

where we have, for convenience, defined

$$\alpha = \frac{(2\pi)^4\,\beta}{8!}\,, \tag{7.41}$$

and where $\hat{X}_{M_1\cdots M_8}$ denotes the components of the 8-form $\hat{X}_{(8)}$, i.e.

$$\hat{X}_{M_1\cdots M_8} = \frac{105}{8(2\pi)^4}\Big(R^{N_1}{}_{N_2[M_1M_2}\, R^{N_2}{}_{|N_3|M_3M_4} R^{N_3}{}_{|N_4|M_5M_6}\, R^{N_4}{}_{|N_1|M_7M_8]} - \\ - \frac{1}{4} R^{N_1}{}_{N_2[M_1M_2}\, R^{N_2}{}_{|N_1|M_3M_4} R^{N_3}{}_{|N_4|M_5M_6}\, R^{N_4}{}_{|N_3|M_7M_8]}\Big)\,. \qquad (7.42)$$

It is convenient also to define

$$\hat{H}^{N_1N_2N_3} \equiv \alpha\, \hat{\epsilon}^{N_1N_2N_3P_1\cdots P_8}\, \hat{X}_{P_1\cdots P_8}\,, \qquad (7.43)$$

so that the field equation (7.40) reads

$$\hat{\nabla}_M\, \hat{F}^{MN_1N_2N_3} = \hat{H}^{N_1N_2N_3}\,. \qquad (7.44)$$

Since we are working only to linear order in β (and hence α), we are allowed to use the zeroth-order background conditions when evaluating $\hat{H}^{N_1N_2N_3}$. We therefore have that the only non-vanishing components of $\hat{H}_{N_1N_2N_3}$ are given by

$$\hat{H}_{0ij} = \alpha\, \epsilon_{ijk_1\cdots k_8}\, X^{k_1\cdots k_8}\,. \qquad (7.45)$$

together with those related by antisymmetry, where $\epsilon_{i_1\cdots i_{10}}$ is the ten-dimensional Levi-Civita tensor.

For a Kähler metric on K_{10}, the Riemann tensor $R_{ijk\ell}$ satisfies

$$R_{ijk\ell} = R_{\hat{i}\hat{j}k\ell} = R_{ij\hat{k}\hat{\ell}}\,. \qquad (7.46)$$

Taking into account the Riemann tensor symmetries, this implies that H_{0ij} given in (7.45) will satisfy

$$H_{0\hat{i}\hat{j}} = H_{0ij}\,. \qquad (7.47)$$

Taking the index value $M=0$ in (7.39) gives

$$\hat{R}_{00}\, \hat{\Gamma}^0\, \hat{\epsilon} - \frac{1}{6} H_{0ij}\, \hat{\Gamma}^{ij}\, \hat{\epsilon} = 0\,. \qquad (7.48)$$

Following the discussion in section 7.1 we may replace the real spinor $\hat{\epsilon}$ by the chiral complex spinor η. Contracting on the left with $\bar{\eta}$, where η is taken to be a Killing spinor, and using its properties as summarised in section 7.1, we deduce that $\hat{R}_{00} = \frac{1}{6} H_{0ij}\, J^{ij}$ and hence that

$$\hat{R}_{00} = \frac{1}{6}\alpha\, J^{ij}\, \epsilon_{ijk_1\cdots k_8}\, X^{k_1\cdots k_8}\,. \qquad (7.49)$$

Taking $M=i$ instead in (7.39), we find after some algebra that

$$\hat{R}_{ij} = \frac{1}{12}\, \alpha\, g_{ij}\, J^{mn}\, \epsilon_{mnk_1\cdots k_8}\, X^{k_1\cdots k_8} - \frac{1}{2}\alpha\, J_i{}^m\, \epsilon_{jmk_1\cdots k_8}\, X^{k_1\cdots k_8}\,. \qquad (7.50)$$

Equations (7.49) and (7.50) represent the gravitational field equations that follow from the integrability conditions for the existence of a Killing spinor. Using (7.28), and now restoring the contribution from the Q_i term in the modified supercovariant derivative, we therefore find that

$$\Box A = \frac{1}{6}\alpha\, J^{ij}\, \epsilon_{ijk_1\cdots k_8}\, X^{k_1\cdots k_8}\,, \tag{7.51}$$

$$\begin{aligned} R_{ij} &= \frac{1}{16}\,\alpha\, g_{ij}\, J^{mn}\, \epsilon_{mnk_1\cdots k_8}\, X^{k_1\cdots k_8} - \frac{1}{2}\alpha\, J_i{}^m\, \epsilon_{jmk_1\cdots k_8}\, X^{k_1\cdots k_8} + \\ &\quad + \frac{\beta}{1152}\,(\nabla_{\hat{i}}\nabla_{\hat{j}}\, Z + \nabla_i\nabla_j\, Z)\,. \end{aligned} \tag{7.52}$$

From the relations between Y_2 and X_8 in a Ricci-flat Kähler manifold, we can show that these equations are identical to (7.32) and (7.33). This establishes consistency, at least, between the bosonic field equations and the conditions that follow from the assumption of supersymmetry persistence in the deformed background.

We now turn to consideration of the full supersymmetry integrability conditions without taking the $\hat{\Gamma}^N$ contraction; these can be read off upon substituting $\hat{F}_4 = G_{(3)} \wedge dt$ into (7.38), and including also the contribution from the Q_i modification. There are two cases to consider: taking the free indices M and N in (7.38) to be either $(MN) = (0i)$ or $(MN) = (ij)$. From $(MN) = (0i)$, we find

$$\nabla_i\nabla_j A\, \Gamma^j\, \eta = -\frac{\mathrm{i}}{18}\, \nabla_i G_{k\ell m}\, \Gamma^{k\ell m}\, \eta\,. \tag{7.53}$$

From this, we find that $G_{(3)}$ is expressible as

$$G_{(3)} = \frac{3}{4} J \wedge dA + \widetilde{G}_{(3)}\,, \tag{7.54}$$

where $\widetilde{G}_{(3)}$ is an arbitrary 3-form that is orthogonal to the Kähler form J, in the sense that

$$J^{jk}\, \widetilde{G}_{ijk} = 0\,. \tag{7.55}$$

From the $(MN) = (ij)$ components of the integrability condition we find, after substituting (7.54), that

$$R_{ijk\ell}\, \Gamma^{k\ell}\, \eta + \frac{3\mathrm{i}}{4}\,(\nabla_i\nabla_{\hat{j}} A - \nabla_j\nabla_{\hat{i}} A)\,\eta + \mathrm{i}\,\nabla_{[i}\widetilde{G}_{j]k\ell}\, \Gamma^{k\ell}\, \eta + \frac{\mathrm{i}\,\beta}{576}(\nabla_i\nabla_{\hat{j}} Z - \nabla_j\nabla_{\hat{i}} Z) = 0\,. \tag{7.56}$$

Multiplying by $\bar{\eta}$, we learn that the Ricci form ϱ_{ij} is given by

$$\varrho_{ij} \equiv \frac{1}{2} R_{ijk\ell}\, J^{k\ell} = -\frac{3}{8}(\nabla_i\nabla_{\hat{j}} A - \nabla_j\nabla_{\hat{i}} A) - \frac{\beta}{1152}(\nabla_i\nabla_{\hat{j}} Z - \nabla_j\nabla_{\hat{i}} Z)\,. \tag{7.57}$$

Multiplying (7.56) instead by $\bar{\eta}\,\Gamma_{mn}$, we obtain two equations, from the real and imaginary parts. The imaginary part yields

$$R_{ij\hat{k}\ell} = -R_{ijk\hat{\ell}} + \nabla_{[i}\widetilde{G}_{j]k\ell} - \nabla_{[i}\widetilde{G}_{j]\hat{k}\hat{\ell}}\,, \tag{7.58}$$

while the real part, after making use of (7.58), again yields (7.57).[12] By making use of the cyclic identity for the Riemann tensor, we can show from (7.58) that

$$R_{i\hat{j}} = -\frac{1}{2} R_{ijk\ell}\, J^{k\ell} - \frac{1}{2}\nabla^k \widetilde{G}_{i\hat{j}\hat{k}} + \frac{1}{2}\nabla^k \widetilde{G}_{ijk}\,. \tag{7.59}$$

Note that the Bianchi identity $d\hat{F}_4 = 0$ implies, from (7.54), that $d\widetilde{G}_{(3)} = 0$, and hence from (7.55) we find that $\nabla^k \widetilde{G}_{i\hat{j}\hat{k}} = 0$, implying that (7.59) reduces to

$$R_{i\hat{j}} = -\frac{1}{2} R_{ijk\ell}\, J^{k\ell} + \frac{1}{2}\nabla^k \widetilde{G}_{ijk}\,. \tag{7.60}$$

Substituting (7.57) into (7.60), and hatting the j index, we obtain the equation

$$R_{ij} = \frac{3}{8}(\nabla_i\nabla_j A + \nabla_{\hat{i}}\nabla_{\hat{j}} A) + \frac{\beta}{1152}(\nabla_i\nabla_j Z + \nabla_{\hat{i}}\nabla_{\hat{j}} Z) - \frac{1}{2}\nabla^k \widetilde{G}_{i\hat{j}\hat{k}}\,. \tag{7.61}$$

In order to verify that our assumption of supersymmetry preservation in the deformed system is consistent, we must show that (7.61) is indeed consistent with the previous expression for the deformed Ricci tensor as given in (7.30), or, equivalently, in (7.52). This can be done by considering the equation of motion for the 4-form field $\hat{F}_{(4)} = G_{(3)} \wedge dt$, namely $\nabla^k G_{ijk} = \alpha\, \epsilon_{ij\ell_1\cdots\ell_8}\, X^{\ell_1\cdots\ell_8}$. Using (7.54), this implies

$$\frac{3}{4} g_{ij}\, \Box A - \frac{3}{4}(\nabla_i\nabla_j A + \nabla_{\hat{i}}\nabla_{\hat{j}} A) + \nabla^k \widetilde{G}_{i\hat{j}\hat{k}} = \alpha\, \epsilon_{\hat{i}\hat{j}\ell_1\cdots\ell_8}\, X^{\ell_1\cdots\ell_8}\,. \tag{7.62}$$

Substituting this into (7.61), we obtain precisely the previous expression (7.52) for the deformed Ricci tensor.

Having verified consistency with the integrability conditions for supersymmetry, it is instructive to examine the supercovariant derivative itself, in the deformed SU(5) holonomy background. In the natural orthonormal frame $\hat{e}^0 = e^A\, dt$, $\hat{e}^i = e^{-\frac{1}{8}A}\, e^i$ for the metric (7.25), we find that to linear order in the $\mathcal{O}(\beta)$ warp function A, the torsion-free spin connection is given by

$$\hat{\omega}_{0i} = -\nabla_i A\, \hat{e}^0\,, \qquad \hat{\omega}_{ij} = \omega_{ij} + \frac{1}{8}(\nabla_i A\, \hat{e}^j - \nabla_j A\, \hat{e}^i)\,, \tag{7.63}$$

and hence from (6.24), with the correction term (3.4) which specialises to (7.37) in the leading-order SU(5) holonomy background, the supercovariant derivative $\hat{D}_A$ in the deformed background is given by

$$\begin{aligned}
\hat{D}_0 &= \partial_0 - \frac{\mathrm{i}}{2}\nabla_i A\, \gamma^i\, \gamma_{11} - \frac{1}{36} G_{ijk}\, \gamma^{ijk}\,,\\
\hat{D}_i &= \nabla_i - \frac{1}{16}\nabla_j A\, \gamma^{ij} + \frac{\mathrm{i}}{72} G_{jk\ell}\, \gamma_i \gamma^{jk\ell}\, \gamma_{11} - \frac{\mathrm{i}}{8} G_{ijk}\, \gamma^{jk}\, \gamma_{11} + \frac{\mathrm{i}\,\beta}{2304}\nabla_{\hat{i}} Z\,,
\end{aligned} \tag{7.64}$$

[12]Equation (7.58) shows that the deformed metric is no longer Kähler (at least with respect to the original Kähler form $J_{ij} = -\mathrm{i}\,\bar{\eta}\Gamma_{ij}\eta$), since if it were, the integrability condition for the covariant constancy of J_{ij}, namely $[\nabla_i, \nabla_j]\, J_{k\ell} = 0$, would imply that $R_{ij\hat{k}\ell} = -R_{ijk\hat{\ell}}$. It is perhaps useful to emphasise here that when looking at the Riemann tensor that arises from the commutation of covariant derivatives, the perturbative scheme in which we are working to order β requires that we must keep terms of order β that represent the deformation away from the leading-order special-holonomy background. By contrast, Riemann tensors appearing in the $\mathcal{O}(\beta)$ correction terms need only be evaluated in the original undeformed special-holonomy background.

when expressed in terms of the ten-dimensional SO(10) Dirac matrices γ_i, and the ten-dimensional chirality operator γ_{11}.

Using these results, we find that the complex spinor $\hat{\eta} = e^{\frac{1}{2}A}\,\eta$ satisfies the $D = 11$ Killing spinor equation $\hat{D}_A\,\hat{\eta} = 0$ provided that η obeys the ten-dimensional equation

$$D_i\eta \equiv \nabla_i\eta + \mathrm{i}\,(\nabla_{\hat{i}} h)\,\eta + \frac{\mathrm{i}}{8}\widetilde{G}_{ijk}\,\gamma^{jk}\eta = 0\,, \tag{7.65}$$

where

$$h = \frac{3}{16}A + \frac{\beta}{2304}\,Z \tag{7.66}$$

together with

$$\gamma_{11}\,\eta = -\eta\,, \qquad \widetilde{G}_{ijk}\,\gamma^{ijk}\,\eta = 0\,. \tag{7.67}$$

As discussed in section 7.1, this result implies the existence of two linearly-independent Majorana Killing spinors of the M-theory background; obtained, in a real representation of the Dirac matrices, by taking the real and imaginary parts of $\hat{\eta}$.

We will now show that this supersymmetry preservation by the M-theory corrections occurs despite a deformation away from SU(5) holonomy. A straightforward calculation from (7.67) shows that $\widetilde{G}_{ijk}$ is the sum of a $(1,2)$ and $(2,1)$ form, with no purely holomorphic or anti-holomorphic $(3,0)$ or $(0,3)$ form components. In other words,

$$(\delta_i^\ell + \mathrm{i}\,J_i{}^\ell)(\delta_j^m + \mathrm{i}\,J_j{}^m)(\delta_k^n + \mathrm{i}\,J_k{}^n)\widetilde{G}_{\ell mn} = 0\,, \tag{7.68}$$

which translates, in the hatted-index notation, into the statement that

$$\widetilde{G}_{ijk} = \widetilde{G}_{i\hat{j}\hat{k}} + \widetilde{G}_{\hat{i}j\hat{k}} + \widetilde{G}_{\hat{i}\hat{j}k}\,. \tag{7.69}$$

Using (7.65), it is straightforward to evaluate $\nabla_j J_i{}^k$ to linear order in the deformation of the metric, where $J_{ij} = -\mathrm{i}\,\bar{\eta}\Gamma_{ij}\eta$, yielding

$$\nabla_j J_i{}^k = \frac{1}{2}\widetilde{G}_{ij}{}^k - \frac{1}{2}\widetilde{G}_{\hat{i}j}{}^{\hat{k}}\,. \tag{7.70}$$

This shows that the loss of Kählerity of the leading-order SU(5) holonomy background is associated with the non-vanishing of the 3-form $\widetilde{G}_{ijk}$. Calculating the Nijenhuis tensor

$$N_{ij}{}^k = \partial_{[j}J_{i]}{}^k - J_i{}^\ell\,J_j{}^k\,\partial_{[m}J_{\ell]}{}^k\,, \tag{7.71}$$

we then find from (7.70) that it is given by

$$N_{ij}{}^k = \frac{1}{2}\left(\widetilde{G}_{ij}{}^k - \widetilde{G}_{i\hat{j}}{}^{\hat{k}} - \widetilde{G}_{\hat{i}j}{}^{\hat{k}} - \widetilde{G}_{\hat{i}\hat{j}}{}^k\right), \tag{7.72}$$

and so from (7.69) we see that the Nijenhuis tensor vanishes. This implies that although the deformed space is no longer Kähler, it is still a complex manifold.

It is worth remarking that although the correction to the SU(5) holonomy background deforms K_{10} into a space that is not only non-Ricci-flat but also non-Kähler, it does have the feature of preserving the vanishing of the first Chern class. This can be seen from the fact that the Ricci form, given by (7.57), is exact.

8. Conclusions

In this paper, we have extended the investigation of string and M-theory corrections to special holonomy backgrounds that was begun in refs. [3, 4, 7] for six-dimensional Calabi-Yau compactifications, and subsequently developed for seven-dimensional G_2 holonomy compactifications in [2]. In the present paper, we have considered the corrections at order α'^3 in string theory for backgrounds of the form $(\text{Minkowski})_2 \times K_8$, where K_8 is a manifold of Spin(7) holonomy. The calculations are considerably more subtle than in the previous cases, because now there are potential contributions to the corrected Einstein equations of a type that would vanish identically by over-antisymmetrisation in the case of curved backgrounds of fewer than eight dimensions. After handling these subtleties, we find that the corrected Einstein equations take a rather simple form, described by (2.28) and (2.30).

We have also considered the structure of the order α'^3 corrections to the supersymmetry transformation rules for an originally Spin(7) holonomy background. Consideration of these corrections is essential if one wants to test whether or not the corrected background remains supersymmetric. We found the simple expression (3.1) for the corrected covariant derivative in the gravitino transformation rule. This expression, which is constructed using the calibrating 4-form of the Spin(7) background, can be recast in a purely riemannian form, where no special tensors existing only in special holonomy backgrounds are needed. Remarkably, the riemannian expression, given in (3.4), turns out to be identical to the one first proposed in [4], whose form was deduced from the (considerably weaker) requirement of supersymmetry preservation for corrected Calabi-Yau six-manifold compactifications. Using the corrected gravitino transformation rule, we illustrated with examples the way in which one can derive corrected first-order equations for metrics that have Spin(7) holonomy at leading order.

We also extended our results to Spin(7) compactifications of M-theory, This was considerably more complicated than the analysis at tree-level in string theory, partly because of the Chern-Simons terms that had to be taken into account and partly because of the topological constraint that forces form fields to become non-vanishing when the Spin(7) manifold is compact (as implied by the term 'compactification'). We gave a complete discussion of the corrections to $(\text{Minkowski})_3 \times K_8$ backgrounds, including for the first time a complete demonstration of supersymmetry preservation in the deformed solutions. Our M-theory result implies a similar result for one-loop corrected Spin(7) compactifications of IIA superstring theory. It would be of interest to extend this to the one-loop corrected IIB superstring theory, but we would not expect this to introduce any essentially new features.

We also considered the case of $(\text{Minkowski})_1 \times K_{10}$ backgrounds in M-theory, where at leading order the manifold K_{10} has a Ricci-flat Kähler metric with SU(5) holonomy. This case is of particular interest because it probes features of M-theory that go beyond those that can be directly accessed from perturbative string theory. In order to avoid the complications arising from a topological constraint, we assumed that $H_8(K_{10})$ is trivial, which implies that K_{10} is non-compact. Under this assumption, we were able to obtain equations for the corrections to the leading-order background. Remarkably, we found that the corrected SU(5) holonomy backgrounds maintain their supersymmetry, assuming only that the previously-known correction term in the gravitino transformation rule plays a rôle. The corrected metric on K_{10} is no longer Kähler, but it is still complex, with vanishing first Chern class. Of course, it would be of considerable interest to extend these results to compact K_{10}.

Finally, we wish to emphasise again the remarkable fact that the form of the correction to the supersymmetry transformation rule first proposed in [4] for string theory in the context of six-dimensional Calabi-Yau compactifications continues to be sufficient to guarantee supersymmetry preservation for compactifications on Spin(7) manifolds. It is also sufficient for Spin(7) compactifications, and certain SU(5) 'non-compactifications' of M-theory. This suggests that it should be taken seriously as a candidate for the complete gravitational part of the string or M-theory correction to the gravitino supersymmetry transformation rule.

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References

[1] D.J. Gross and E. Witten, *Superstring modifications of Einstein's equations*, *Nucl. Phys.* **B 277** (1986) 1.

[2] H. Lü, C.N. Pope, K.S. Stelle and P.K. Townsend, *Supersymmetric deformations of G_2 manifolds from higher-order corrections to string and M-theory*, *JHEP* **10** (2004) 019 [`hep-th/0312002`].

[3] M.D. Freeman and C.N. Pope, *Beta functions and superstring compactifications*, *Phys. Lett.* **B 174** (1986) 48.

[4] P. Candelas, M.D. Freeman, C.N. Pope, M.F. Sohnius and K.S. Stelle, *Higher order corrections to supersymmetry and compactifications of the heterotic string*, *Phys. Lett.* **B 177** (1986) 341.

[5] M.D. Freeman, C.N. Pope, M.F. Sohnius and K.S. Stelle, *Supersymmetry in compactifications of the heterotic string*, in Paris-Meudon Colloq. (1986) 271.

[6] D. Constantin, *M-theory vacua from warped compactifications on* Spin(7) *manifolds*, *Nucl. Phys.* **B 706** (2005) 221 [`hep-th/0410157`].

[7] H. Lü, C.N. Pope and K.S. Stelle, *Higher-order corrections to non-compact Calabi-Yau manifolds in string theory*, *JHEP* **07** (2004) 072 [`hep-th/0311018`].

[8] C.N. Pope, M.F. Sohnius and K.S. Stelle, *Counterterm counterexamples*, *Nucl. Phys.* **B 283** (1987) 192.

JHEP07(2005)075

[9] M. Cvetič, G.W. Gibbons, H. Lü and C.N. Pope, *Cohomogeneity one manifolds of* Spin(7) *and G_2 holonomy*, *Phys. Rev.* **D 65** (2002) 106004 [hep-th/0108245].

[10] R.L. Bryant and S. Salamon, *On the construction of some complete metrics with exceptional holonomy*, *Duke Math. J.* **58** (1989) 829.

[11] G.W. Gibbons, D.N. Page and C.N. Pope, *Einstein metrics on S^3 R^3 and R^4 bundles*, *Commun. Math. Phys.* **127** (1990) 529.

[12] M. Cvetič, G.W. Gibbons, H. Lü and C.N. Pope, *New complete non-compact* Spin(7) *manifolds*, *Nucl. Phys.* **B 620** (2002) 29 [hep-th/0103155]; *New cohomogeneity one metrics with* Spin(7) *holonomy*, *J. Geom. Phys.* **49** (2004) 350 [math.DG/0105119].

[13] S. Gukov and J. Sparks, *M-theory on* Spin(7) *manifolds, I*, *Nucl. Phys.* **B 625** (2002) 3 [hep-th/0109025].

[14] K. Becker and M. Becker, *M-theory on eight-manifolds*, *Nucl. Phys.* **B 477** (1996) 155 [hep-th/9605053].

[15] K. Becker and M. Becker, *Supersymmetry breaking, M-theory and fluxes*, *JHEP* **07** (2001) 038 [hep-th/0107044].

[16] S.W. Hawking and M.M. Taylor-Robinson, *Bulk charges in eleven dimensions*, *Phys. Rev.* **D 58** (1998) 025006 [hep-th/9711042].

[17] K. Becker, *A note on compactifications on* Spin(7)*-holonomy manifolds*, *JHEP* **05** (2001) 003 [hep-th/0011114].

[18] F.A. Brito, M. Cvetič and A. Naqvi, *Brane resolution and gravitational Chern-Simons terms*, *Class. and Quant. Grav.* **20** (2003) 285 [hep-th/0206180].

[19] C.J. Isham and C.N. Pope, *Nowhere vanishing spinors and topological obstructions to the equivalence of the NSR and GS superstrings*, *Class. and Quant. Grav.* **5** (1988) 257.

[20] C.J. Isham, C.N. Pope and N.P. Warner, *Nowhere vanishing spinors and triality rotations in eight manifolds*, *Class. and Quant. Grav.* **5** (1988) 1297.

[21] M.J. Duff and K.S. Stelle, *Multi-membrane solutions of $D = 11$ supergravity*, *Phys. Lett.* **B 253** (1991) 113.

[22] N. Deruelle and L. Farina-Busto, *The Lovelock gravitational field equations in cosmology*, *Phys. Rev.* **D 41** (1990) 3696.

[23] K. Peeters, P. Vanhove and A. Westerberg, *Supersymmetric higher-derivative actions in ten and eleven dimensions, the associated superalgebras and their formulation in superspace*, *Class. and Quant. Grav.* **18** (2001) 843 [hep-th/0010167].

[24] D.D. Joyce, *Compact manifolds with special holonomy*, Oxford University Press, Oxford 2000.

[25] M.B. Green and P. Vanhove, *D-instantons, strings and M-theory*, *Phys. Lett.* **B 408** (1997) 122 [hep-th/9704145].

[26] M. Cvetič, H. Lü and C.N. Pope, *Brane resolution through transgression*, *Nucl. Phys.* **B 600** (2001) 103 [hep-th/0011023].

[27] M. Cvetič, G.W. Gibbons, H. Lü and C.N. Pope, *New complete non-compact* Spin(7) *manifolds*, *Nucl. Phys.* **B 620** (2002) 29 [hep-th/0103155].